# A Study of a Linear Ordinary Second Order Differential Equation With Five Regular Singular Points

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This Dissertation is affectionately

dedicated to my wife

Betty

## PREFACE

In this dissertation, certain general aspects of linear ordinary second order differential equations are initially discussed. Attention is briefly devoted to establishing criteria under which it is possible to discern whether or not the point at infinity is an ordinary point or a regular singular point of a given differential equation. This is followed by a derivation of the canonical form of a differential equation with three regular singular points. Certain constants in this equation are specialized, and the result is an equation with two regular singular points. A closed form solution of this latter equation is then given.

Hypergeometric functions are deduced from a study of a solution of the hypergeometric equation. Criteria for convergence of hypergeometric functions are also established.

In Chapter II, an equation with five elementary singular points is introduced. This is followed by a derivation and brief discussion of the respective equations of Lamé, Mathieu, Legendre, Bessel, Weber, and Stokes.

In Chapter III, an equation with n-regular singular points is discussed, and certain equations are derived which are used in Chapters IV and V. A brief discussion of Heun's equation is given in Chapter IV, which includes its derivation and solution in a general case.

A derivation of a general differential equation, with five non-elementary regular singular points, is given in Chapter V. Information gained from Chapters III and IV prove useful in this derivation. This general differential equation is then solved in a general case in Chapter VI.

An exposition of Scheffé's criteria, as applied to a linear ordinary second order differential equation, is given in Chapter VII. These criteria are then applied to the equation derived in Chapter V. The result is Chapter VIII. In this chapter, solutions in a neighborhood of each of the regular singular points $z = 0$, 1, a, b, and infinity are given in terms of hypergeometric functions. A solution in a neighborhood of $z = d$, where d is an ordinary point, is also expressed in terms of hypergeometric functions. Chapter VIII contains certain confluent equations which are classified in the notation of Ince.

The majority of Chapters V, VI, and VIII is believed to be new. The other chapters, while not original, are nevertheless considered necessary for a logical development of the study.

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Henry L. Crowson

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# CHAPTER I

## INTRODUCTION

### Some General Considerations of Linear Ordinary Differential Equations

A linear ordinary second order differential equation is frequently written in the form:

$$(1) \qquad \frac{d^2u}{dz^2} + p(z)\frac{du}{dz} + q(z)u = 0.$$

We assume u and z to be complex variables with u designated as the dependent variable and z the independent variable. We further assume the existence of a closed region S in the complex plane throughout which both p(z) and q(z) are analytic [14, p. 83 ] except at a finite number of poles [7, p. 38 ].[1]

If p(z) and q(z) are both analytic at some point z = c in the region S, then z = c is designated as an ordinary point of S. Other points in S are designated as singular points. Singular points of p(z) and q(z) are either regular singular points or irregular singular points.

We define a point z = c, which is in S and not on the boundary of S, to be a finite regular singular point of the differential equation (1) if both $(z-c)p(z)$ and $(z-c)^2q(z)$ are analytic at c. From this definition it

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[1]Numbers in brackets refer to the bibliography.

follows that $p(z)$ has a simple pole at $c$, and $q(z)$ has a pole of order 2 at $c$. That is, we can represent $p(z)$ and $q(z)$ by the following series expansion:

$$(2) \qquad p(z) = \frac{p_{-1}}{z-c} + p_0 + p_1(z-c) + \cdots + p_n(z-c)^n + \cdots ,$$

and

$$(3) \qquad q(z) = \frac{q_{-2}}{(z-c)^2} + \frac{q_{-1}}{z-c} + q_0 + q_1(z-c) + \cdots + q_n(z-c)^n + \cdots,$$

where $p_n$, $(n = -1,0,1,2,\ldots)$ and $q_n$, $(n = -2,-1,0,1,2,\ldots)$ are constants, and not all of the quantities $p_{-1}$, $q_{-2}$, and $q_{-1}$ can be zero.

We now define

$$(4) \qquad P(z-c) = (z-c)\cdot p(z), \text{ and}$$

$$(5) \qquad Q(z-c) = (z-c)^2 q(z), \text{ so that from equations (2) and (3) we get:}$$

$$(6) \qquad P(z-c) = p_{-1} + p_0(z-c) + p_1(z-c)^2 + \cdots + p_n(z-c)^{n+1} + \cdots , \text{ and}$$

$$(7) \qquad Q(z-c) = q_{-2} + q_{-1}(z-c) + q_0(z-c)^2 + \cdots + q_n(z-c)^{n+2} + \cdots .$$

In summation notation, equations (6) and (7) become:

$$(8) \qquad P(z-c) = \sum_{j=0}^{\infty} (P_j)(z-c)^j, \text{ and}$$

$$(9) \qquad Q(z-c) = \sum_{j=0}^{\infty} (Q_j)(z-c)^j, \text{ where } P_j \text{ and } Q_j \text{ are constants.}$$

From equations (6) - (9) we note that $P(z-c)$ and $Q(z-c)$ are

both analytic at $z = c$, and further that $P_j = p_{j-1}$, and $Q_j = q_{j-2}$.

We may substitute equations (4) and (5) into equation (1), and rearrange slightly to obtain:

$$(10) \qquad (z-c)^2 \frac{d^2u}{dz^2} + (z-c)P(z-c)\frac{du}{dz} + Q(z-c)u = 0.$$

We will now investigate the conditions under which infinity is either an ordinary point or a regular singular point. Initially we make the variable transformation

(11) $z = 1/w$, following which we will investigate the behavior of $p(z)$ and $q(z)$ in a neighborhood of $w = 0$.

We note the following equations:

$$(12) \qquad \frac{du}{dz} = \frac{du}{dw} \cdot \frac{dw}{dz} \text{ , hence from equation (11)}$$

$$(13) \qquad \frac{du}{dz} = -\frac{1}{z^2} \cdot \frac{du}{dw} \text{ . Also from equation (12) we have:}$$

$$(14) \qquad \frac{d^2u}{dz^2} = \frac{du}{dw} \cdot \frac{d^2w}{dz^2} + \left(\frac{dw}{dz}\right)^2 \cdot \frac{d^2u}{dw^2} \text{ . From equation (11)}$$

we now obtain:

$$(15) \qquad \frac{d^2u}{dz^2} = \frac{2}{z^3} \cdot \frac{du}{dw} + \frac{1}{z^4} \cdot \frac{d^2u}{dw^2} \text{ .}$$

Substituting equations (13) and (15) into equation (1) yields the following expression:

(16) $$w^4 \frac{d^2u}{dw^2} + 2w^3 \frac{du}{dw} - p(1/w)[w^2 \frac{du}{dw}] + q(1/w)u = 0.$$

Simplifying equation (16) yields:

(17) $$\frac{d^2u}{dw^2} + [2/w - \frac{1}{w^2} p(1/w)]\frac{du}{dw} + \frac{1}{w^4} q(1/w)u = 0.$$

Now $w = 0$ will be an ordinary point of equation (17) if both $[2/w - \frac{1}{w^2} p(1/w)]$ and $\frac{1}{w^4} q(1/w)$ are analytic at $w = 0$.

The condition $[2/w - \frac{1}{w^2} p(1/w)]$ be analytic at $w = 0$ will be met if $p(1/w)$ is expressed as follows:

(18) $$p(1/w) = 2w + p_2w^2 + \cdots + p_nw^n + \cdots .$$

From equation (18) it follows that

(19) $$[2/w - \frac{1}{w^2} p(1/w)] = - p_2 - p_3w - p_4w^2 - \cdots - p_nw^{n-2} - \cdots .$$

This is clearly analytic at $w = 0$ for finite $p_2$, where $p_n$, $(n = 2,3,\cdots)$ are constants.

Further, the condition $\frac{1}{w^4} q(1/w)$ be analytic at $w = 0$ will be met if $q(1/w)$ is expressed as follows:

(20) $$q(1/w) = q_4w^4 + q_5w^5 + \cdots + q_nw^n + \cdots; \quad q_4 \neq 0.$$

From equation (20) we obtain:

(21) $$\frac{1}{w^4} q(1/w) = q_4 + q_5w + \cdots + q_nw^{n-4} + \cdots , \text{ where}$$

$q_n$, $(n = 4,5\ldots)$ are constants. By saying an equation is analytic at a point, we will mean the function defined by the equation is analytic at that point. Thus, equation (21) is analytic at $w = 0$.

Substituting equation (11) into equations (18) and (20), we conclude that infinity will be an ordinary point of equation (1) provided

(22) $\quad p(z) = 2/z + p_2/z^2 + \cdots + p_n/z^n + \cdots$, and

(23) $\quad q(z) = q_4/z^4 + q_5/z^5 + \cdots + q_n/z^n + \cdots$.

We will now investigate the conditions under which infinity is a regular singular point.

From equation (17), if it is true that both $w[2/w - \frac{1}{w^2}\, p(1/w)]$ and $\frac{w^2 q(1/w)}{w^4}$ are analytic at $w = 0$, and that either $[2/w - \frac{1}{w^2}\, p(1/w)]$ or $\frac{q(1/w)}{w^4}$ or both fail to be analytic at $w = 0$, then $w = 0$ is a regular singular point of equation (17). We will use this definition consistently. This implies, from equation (11), that infinity is a regular singular point of equation (1).

Assume an expansion for $p(1/w)$ of the form:

(24) $\quad p(1/w) = p_1 w + p_2 w^2 + \cdots + p_n w^n + \cdots$. Then

(25) $\quad w[2/w - \frac{1}{w^2}\, p(1/w)] = 2 - p_1 - p_2 w - \cdots - p_n w^{n-1} - \cdots$.

Equation (25) is clearly analytic at $w = 0$, for finite $p_1$, which implies that

(26) $zp(z) = p_1 + p_2/z + \cdots + p_n/z^{n-1} + \cdots$ is analytic at infinity. Again we note that $p_n$, $(n = 1,2,\ldots)$, are constants.

We next consider $\frac{w^2q(1/w)}{w^4}$, by first assuming an expansion of $q(1/w)$ of the form:

(27) $q(1/w) = q_2w^2 + q_3w^3 + \cdots + q_nw^n + \cdots$, where not all of the quantities $p_1$, $q_2$, and $q_3$ are zero.

Then from equation (27) we get:

(28) $$\frac{w^2q(1/w)}{w^4} = q_2 + q_3w + \cdots + q_nw^{n-2} + \cdots .$$

Equation (28) is evidently analytic at $w = 0$, so from equations (11) and (28) we conclude that

(29) $z^2q(z) = q_2 + q_3/z + \cdots + q_n/z^{n-2} + \cdots$ is analytic at infinity.

We may summarize the above statements concerning ordinary points and regular singular points of equation (1) as follows:

$z=c$ is a finite ordinary point if both $p(z)$ and $q(z)$ are analytic at $z = c$. $z = c$ is a finite regular singular point if both $(z-c)p(z)$ and $(z-c)^2q(z)$ are analytic at $z = c$, and if either $p(z)$ or $q(z)$ or both fail to be analytic at $z = c$.

Next, let $z = 1/w$, then consider the expressions $[2/w - \frac{1}{w^2}p(1/w)]$ and $\frac{q(1/w)}{w^4}$. If both of these expressions are analytic at $w = 0$, then $w = 0$ is an ordinary point of equation (17). This implies that infinity is an ordinary point of equation (1). Again we consider the expressions $w[2/w - \frac{1}{w^2}\, p(1/w)]$ and $\frac{w^2 q(1/w)}{w^4}$. If both of these expressions are analytic at $w = 0$, then $w = 0$ is a regular singular point of equation (17). This implies that infinity is a regular singular point of equation (1).

Points in the extended z-plane which do not conform to the above criteria are designated as irregular singular points of equation (1).

To further illustrate the conditions under which infinity is either an ordinary point or a regular singular point, we give a concrete example. Thus, let $p(z)$ in equation (1) be a rational function of the form:

$$(30) \qquad p(z) = \frac{\sum_{i=0}^{k} A_i z^{k-i}}{\sum_{i=0}^{m} B_i z^{m-i}} \; ; \text{ where } (A_o)(B_o) \neq 0.$$

Dividing numerator and denominator of equation (30) by $\sum_{i=0}^{m} B_i z^{m-i}$ gives us:

$$(31) \qquad p(z) = (A_o/B_o)z^{k-m} + \cdots .$$

Again from equation (1), let q(z) be a rational function of the form:

$$(32) \qquad q(z) = \frac{\sum_{i=0}^{n} C_i z^{n-i}}{\sum_{i=0}^{m} B_o z^{m-i}}; \quad \text{where } C_o \neq 0.$$

We may rewrite equation (32) thus:

$$(33) \qquad q(z) = (C_o/B_o)z^{n-m} + \cdots .$$

Now infinity will be an ordinary point of equation (1) if both p(z) and q(z) are analytic at infinity. These conditions are fulfilled, from equations (31) and (33), if both $k-m \leq 0$ and $n-m \leq 0$.

Again from equations (31) and (33),

$$(34) \qquad zp(z) = (A_o/B_o)z^{k-m+1} + \cdots , \text{ and}$$

$$(35) \qquad z^2q(z) = (C_o/B_o)z^{n-m+2} + \cdots .$$

Thus infinity is a regular singular point of equation (1) if both zp(z) and $z^2q(z)$ are analytic at infinity. These conditions are met, from equations (34) and (35) if both $k-m+1 \leq 0$ and $n-m+2 \leq 0$, and if either p(z) or q(z) or both fail to be analytic at infinity.

The above fundamental considerations, while not extensive, are nonetheless pertinent to the future development of this study. Other applicable topics are developed as the study progresses.

We will now apply the foregoing theory to a discussion of a differential equation with three regular singular points.

## Differential Equations with Three Regular Singular Points

Let equation (1) have three finite regular singular points $z = a$, $z = b$ and $z = c$. We assume that equation (1) has no other singular points, and that the points a, b, and c, have respective exponents $\alpha$, $\alpha'$; $\beta$, $\beta'$; and $\gamma$, $\gamma'$. Then $p(z)$ is a rational function with simple poles at a, b, and c, and $q(z)$ is a rational function with poles of order 2 at each of a, b, and c.

We may now write $p(z)$ as follows:

$$(36) \qquad p(z) = \frac{E_2 z^2 + E_1 z + E_o}{(z-a)(z-b)(z-c)} \text{ , or expanding equation (36)}$$

in partial fractions yields:

$$(37) \qquad p(z) = G_1/(z-a) + G_2/(z-b) + G_3/(z-c).$$

We may write an expression similar to equation (36) for $q(z)$, thus obtaining:

$$(38) \qquad q(z) = \frac{F_2 z^2 + F_1 z + F_o}{(z-a)(z-b)(z-c)} \cdot \frac{1}{(z-a)(z-b)(z-c)} .$$

We may expand equation (38) in partial fraction to get:

$$(39) \qquad q(z) = [H_1/(z-a) + H_2/(z-b) + H_3/(z-c)]$$

$$\cdot[1/(z-a)(z-b)(z-c)].$$

Substituting equations (37) and (39) into equation (1) gives us the following expression:

$$(40) \qquad \frac{d^2u}{dz^2} + [G_1/(z-a) + G_2/(z-b) + G_3/(z-c)]\frac{du}{dz}$$

$$+ [H_1/(z-a) + H_2/(z-b) + H_3/(z-c)]$$

$$\cdot[1/(z-a)(z-b)(z-c)]u = 0.$$

We now assume a solution of equation (40) in a neighborhood of $z = a$ to be of the form:

$$(41) \qquad u(z) = c_0(z-a)^{\lambda} + \sum_{r=1}^{\infty} c_r(z-a)^{\lambda+r}; \quad c_0 \neq 0.$$

Differentiating equation (41) with respect to $(z-a)$ yields:

$$(42) \qquad u'(z) = c_0\lambda(z-a)^{\lambda-1} + \sum_{r=1}^{\infty} c_r(\lambda+r)(z-a)^{\lambda+r-1},$$

where it is understood that $u'(z) = \frac{du}{d(z-a)}$.

Differentiating equation (42) with respect to $(z-a)$ yields the following expression:

$$(43) \qquad u''(z) = c_0\lambda(\lambda-1)(z-a)^{\lambda-2} + \sum_{r=1}^{\infty} c_r(\lambda+r)(\lambda+r-1)(z-a)^{\lambda+r-2}.$$

We now substitute equations (41), (42) and (43) into equation (40) to get:

$$(44)\quad c_0\lambda(\lambda-1)(z-a)^{\lambda-2} + \sum_{r=1}^{\infty} c_r(\lambda+r)(\lambda+r-1)(z-a)^{\lambda+r-2}$$
$$+ \left[\frac{G_1}{z-a} + \frac{G_2}{z-b} + \frac{G_3}{z-c}\right] \cdot \left[c_0\lambda(z-a)^{\lambda-1} + \sum_{r=1}^{\infty} c_r(\lambda+r)(z-a)^{\lambda+r-1}\right]$$
$$+ \left[\frac{H_1/(z-a) + H_2/(z-b) + H_3/(z-c)}{(z-a)(z-b)(z-c)}\right]$$
$$\cdot\left[c_0(z-a)^{\lambda} + \sum_{r=1}^{\infty} c_r(z-a)^{\lambda+r}\right] = 0.$$

Inasmuch as we are expanding in ascending powers of $(z-a)$, we equate the coefficient of $(z-a)^{\lambda-2}$ to zero thus obtaining the indicial equation:

(45) $c_0\lambda(\lambda-1) + c_0G_1\lambda + c_0k = 0$, where k is to be determined from the coefficient of $u(z)$ in equation (44). We will now compute k.

We write equation (39) as follows:

$$(46)\quad q(z) = \frac{H_1}{(z-a)^2}\cdot\frac{1}{(z-b)(z-c)} + \left[\frac{H_2}{z-b} + \frac{H_3}{z-c}\right]\left[\frac{1}{(z-a)(z-b)(z-c)}\right].$$

Let

(47) $f(z) = \frac{1}{(z-b)(z-c)}$, and expand in a Taylor's series about $z = a$ to get:

$$(48) \qquad f(z) = f(a) + \sum_{n=1}^{\infty} \frac{(z-a)^n f^{(n)}(a)}{n!} .$$

Substitute equation (48) into equation (46) to get:

$$(49) \qquad q(z) = \frac{H_1 f(a)}{(z-a)^2} + H_1 \sum_{n=1}^{\infty} \frac{(z-a)^{n-2} f^{(n)}(a)}{n!}$$

$$+ [H_2/(z-b) + H_3/(z-c)][1/(z-a)(z-b)(z-c)].$$

Substituting equation (49) into (44), where f(a) is computed from equation (47), gives the following expression:

$$(50) \quad c_0\lambda(\lambda-1)(z-a)^{\lambda-2} + \sum_{r=1}^{\infty} c_r(\lambda+r)(\lambda+r-1)(z-a)^{\lambda+r-2}$$

$$+ \left[\frac{G_1}{z-a} + \frac{G_2}{z-b} + \frac{G_3}{z-c}\right]\left[c_0\lambda(z-a)^{\lambda-1} + \sum_{r=1}^{\infty} c_r(\lambda+r)(z-a)^{\lambda+r-1}\right]$$

$$+ \{H_1/(a-b)(a-c)(z-a)^2 + H_1 \sum_{n=1}^{\infty} \frac{(z-a)^{n-2} f^{(n)}(a)}{n!}$$

$$+ [H_2/(z-b) + H_3/(z-c)][1/(z-a)(z-b)(z-c)]\}\{c_0(z-a)^{\lambda}$$

$$+ \sum_{r=1}^{\infty} c_r(z-a)^{\lambda+r}]\} = 0.$$

The indicial equation is now complete. By inspection of equation (50), we note that

(51) $k = H_1/(a-b)(a-c)$. Substitution into equation (45) and then division of both members by $c_0$ gives, upon slight rearrangement:

$$(52) \qquad \lambda^2 + \lambda(G_1 - 1) + H_1/(a-b)(a-c) = 0.$$

The roots of equation (52) are the exponents of the solution to equation (40) in a neighborhood of $z = a$. Hence from equation (52) we immediately deduce

(53) $\quad 1 - G_1 = \alpha + \alpha'$, and

(54) $\quad H_1/(a-b)(a-c) = \alpha\alpha'$.

Solving equations (53) and (54) we have:

(55) $\quad G_1 = 1 - \alpha - \alpha'$, and

(56) $\quad H_1 = \alpha\alpha'(a-b)(a-c)$.

Similarly, solving equation (40) in a neighborhood of $z = b$ and $z = c$, respectively, yields:

(57) $\quad G_2 = 1 - \beta - \beta'$,

(58) $\quad H_2 = \beta\beta'(b-a)(b-c)$,

(59) $\quad G_3 = 1 - \gamma - \gamma'$, and

(60) $\quad H_3 = \gamma\gamma'(c-a)(c-b)$.

Substituting equations (55) - (60) into equation (40) we obtain:

$$(61) \quad \frac{d^2u}{dz^2} + \left[\frac{1-\alpha-\alpha'}{z-a} + \frac{1-\beta-\beta'}{z-b} + \frac{1-\gamma-\gamma'}{z-c}\right]\frac{du}{dz}$$

$$+ \left[\frac{\alpha\alpha'(a-b)(a-c)}{z-a} + \frac{\beta\beta'(b-a)(b-c)}{z-b}\right.$$

$$\left. + \frac{\gamma\gamma'(c-a)(c-b)}{z-c}\right] \cdot \left[\frac{u}{(z-a)(z-b)(z-c)}\right] = 0.$$

Equation (61) is the canonical form of a second order linear ordinary differential equation with three finite regular singular points.

This canonical form is due to Papperitz [10, p. 213].

We note that equation (61) is determined completely and uniquely by the singular points and the exponents at the singular points.

Riemann [12, vol. 7] expressed a solution to equation (61) as follows:

$$(62) \qquad u = P \left\{ \begin{matrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{matrix} \right\}.$$

The singular points are in the first row, with corresponding exponents, at the respective singular points, in the second and third rows. The independent variable is in the fourth column.

Equation (62) is called Riemann's P-equation.

We will now specialize certain of the restrictions attached to equation (61) and hence derive a differential equation with two regular singular points. This equation will then be solved in closed form.

## Differential Equations with Two Regular Singular Points

In equation (61) we let $z = c$ be an ordinary point. Then both $p(z)$ and $q(z)$ are analytic at $c$. It then follows that

(63) $\quad 1 - \gamma - \gamma' = 0$ and

(64) $\quad \gamma\gamma' = 0$, for otherwise $z = c$ is a regular singular point. Equation (61) now becomes:

$$(65)\quad \frac{d^2u}{dz^2} + [(1-\alpha-\alpha')/(z-a) + (1-\beta-\beta')/(z-b)]\frac{du}{dz}$$

$$+ [\alpha\alpha'(a-b)(a-c)/(z-a) + \beta\beta'(b-c)(b-a)/(z-b)]$$

$$\cdot[u/(z-a)(z-b)(z-c)] = 0.$$

Now in order for $z = c$ to be an ordinary point $z - c$ must divide the coefficient of $u/(z-a)(z-b)(z-c)$. That is,

$$(66)\quad Q(z) = \frac{\alpha\alpha'(a-b)(a-c)(z-b) + \beta\beta'(b-c)(b-a)(z-a)}{(z-a)(z-b)}$$

must be divisible by $z - c$, where $Q(z) = (z-a)(z-b)(z-c)q(z)$.

From equation (66) we deduce that

$$Q(c) = \frac{\alpha\alpha'(a-b)(a-c)(c-b) + \beta\beta'(b-c)(b-a)(c-a)}{(c-a)(c-b)} = 0,$$

and hence we get:

(67) $\quad \alpha\alpha' = \beta\beta'$.

In Chapter III we will show that for a linear ordinary second order differential equation with regular singular points $a_r$, $(r = 1,2,\ldots,n)$, and exponents at these singular points of $\alpha_r$ and $\beta_r$, $(r = 1,2,\ldots,n)$, then it is true that

$$\sum_{r=1}^{n} (\alpha_r + \beta_r) = n-2.$$

Applying this expression to equation (61) we note that

(68) $\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1$, but from equation (63) this simplifies to:

(69) $\alpha + \alpha' + \beta + \beta' = 0$. We will use this relationship presently.

Now substitute equation (67) into equation (66), then substitute the result into equation (65) to get:

$$(70)\quad \frac{d^2u}{dz^2} + [(1-\alpha-\alpha')/(z-a) + (1-\beta-\beta')/(z-b)]\frac{du}{dz} + \frac{\alpha\alpha'[(a-b)(a-c)(z-b) + (b-c)(b-a)(z-a)]u}{(z-a)^2(z-b)^2(z-c)} = 0.$$

Equation (70) simplifies to:

$$(71)\quad \frac{d^2u}{dz^2} + [(1-\alpha-\alpha')/(z-a) + (1-\beta-\beta')/(z-b)]\frac{du}{dz} + [\alpha\alpha'(a-b)^2/(z-a)^2(z-b)^2]u = 0.$$

In order to solve equation (71) we first make the variable transformation:

(72) $w = (z-a)/(z-b)$ from which it follows that

(73) $z = (bw - a)/(w-1)$,

(74) $z-a = w(b-a)/(w-1)$, and

(75) $z-b = (b-a)/(w-1)$.

Now from equations (12), (14) and (72) we get:

(76) $$\frac{du}{dz} = \frac{du}{dw}[(w-1)^2/(a-b)], \text{ and}$$

(77) $$\frac{d^2u}{dz^2} = \frac{du}{dw}[2(w-1)^3/(a-b)^2] + \frac{d^2u}{dw^2}[(w-1)^4/(a-b)^2].$$

Substituting equations (74) - (77) into equation (72) yields the following equation:

(78) $$\frac{du}{dw}[2(w-1)^3/(a-b)^2] + \frac{d^2u}{dw^2}[(w-1)^4/(a-b)^2]$$

$$+ \left[\frac{(1-\alpha-\alpha')(w-1)}{w(b-a)} + \frac{(1-\beta-\beta')(w-1)}{(b-a)}\right][(w-1)^2/(a-b)]\frac{du}{dw}$$

$$+ \alpha\alpha' \frac{(a-b)^2(w-1)^4}{w^2(b-a)^4} u = 0.$$

We now rearrange equation (78) as follows:

(79) $$\frac{(w-1)^4}{(a-b)^2}\cdot\frac{d^2u}{dw^2} - \frac{(w-1)^3}{(a-b)^2}[(1-\alpha-\alpha')/w + (1-\beta-\beta')]\frac{du}{dw}$$

$$+ \frac{2(w-1)^3}{(a-b)^2}\cdot\frac{du}{dw} + \frac{(w-1)^4}{w^2(a-b)^2}\cdot(\alpha\alpha')u = 0.$$

Dividing both members of equation (79) by $\frac{(w-1)^4}{(a-b)^2}$ we obtain:

$$(80)\qquad \frac{d^2u}{dw^2} - [1/(w-1)]\cdot[(1-\alpha-\alpha')/w + (1-\beta-\beta')]\frac{du}{dw} + [2/(w-1)]\frac{du}{dw} + (\alpha\alpha'/w^2)u = 0.$$

We simplify equation (80) further as follows:

$$(81)\qquad \frac{d^2u}{dw^2} - \frac{1}{w(w-1)}[(1-\alpha-\alpha') + w(1-\beta-\beta') - 2w]\frac{du}{dw} + (\alpha\alpha'/w^2)u = 0.$$

Multiplying both members of equation (81) by $w^2(w-1)$ to get:

$$(82)\qquad w^2(w-1)\frac{d^2u}{dw^2} + [w^2(1+\beta+\beta') - w(1-\alpha-\alpha')]\frac{du}{dw} + \alpha\alpha'(w-1)u = 0.$$

We now solve equations (67) and (69) simultaneously to get $\beta = -\alpha$, and $\beta' = -\alpha'$, hence equation (82) becomes:

$$(83)\qquad w^2(w-1)\frac{d^2u}{dw^2} + [(w^2-w)(1-\alpha-\alpha')]\frac{du}{dw} + \alpha\alpha'(w-1)u = 0.$$

Since we are assuming a solution in the finite plane, division by (w-1) is permissible, and equation (83) becomes:

(84) $w^2 \frac{d^2u}{dw^2} + [(1-\alpha-\alpha')(w)]\frac{du}{dw} + \alpha\alpha' u = 0.$

Let $D_w \equiv \frac{d}{dw}$ , then equation (84) becomes:

(85) $[w^2 D_w^2 + (1-\alpha-\alpha')wD_w + \alpha\alpha']u = 0.$

We now let

(86) $w = e^x$, then equation (85) becomes

(87) $[D_x(D_x-1) + (1-\alpha-\alpha')D_x + \alpha\alpha']u = 0$, where

$D_x \equiv \frac{d}{dx}$ .

Upon simplification, equation (87) becomes:

(88) $[(D_x - \alpha)(D_x - \alpha')]u = 0.$

A general solution of equation (88) is:

(89) $u = K_1 e^{\alpha x} + K_2 e^{\alpha' x}$, where $K_1$ and $K_2$ are arbitrary constants of integration. [8, p. 18]

(90) Now $w^\alpha = e^{\alpha x}$ and $w^{\alpha'} = e^{\alpha' x}$, hence combining equation (90) and (72), we see that equation (89) becomes:

(91) $u(z) = K_1[(z-a)/(z-b)]^\alpha + K_2[(z-a)/(z-b)]^{\alpha'}.$

Equation (91) is the desired solution of equation (71).

Now if we suppose $\alpha = \alpha'$, then equation (88) becomes:

(92) $(D_x - \alpha)^2 u = 0$, and we assume a solution of the form:

(93) $u = te^{\alpha x}$, where t and x are complex variables. From equation (93) we will determine t such that equation (93) is a general solution of equation (85) with $\alpha = \alpha'$.

Substituting equation (93) into equation (92) we have:

(94) $(D_x-\alpha)^2(e^{\alpha x}\cdot t) = e^{\alpha x}D_x^2 t.$

(95) Now if $D_x^2 t = 0$, then $e^{\alpha x}\cdot t$ will satisfy equation (92).

Solving equation (95) we get:

(96) $t = K_3 + K_4 x$, and hence from equation (93),

(97) $u = K_3 e^{\alpha x} + K_4 x e^{\alpha x}.$

From equation (86) we have

$x = \ln w$, and substitution into equation (97) gives us the following:

$u = K_3 e^{\alpha \ln w} + K_4 \ln w e^{\alpha \ln w}$, so that

(98) $u = K_3 w^{\alpha} + K_4 w^{\alpha} \ln w.$

Substituting equation (72) into equation (98) gives us the final result:

(99) $$u(z) = K_3\left[\frac{z-a}{z-b}\right]^{\alpha} + K_4\left[\frac{z-a}{z-b}\right]^{\alpha} \ln\left[\frac{z-a}{z-b}\right].$$

Equation (99) is a solution in closed form of a linear ordinary second order differential equation with two finite regular singular points.

We will now discuss a particular differential equation from which we will derive the hypergeometric series, and then define the hypergeometric function.

## Hypergeometric Functions

We will now solve a particular differential equation, thus:

$$(100) \qquad \frac{d^2u}{dz^2} + \frac{[c - (a+b+1)z]}{z(1-z)}\frac{du}{dz} - \frac{ab}{z(1-z)}u = 0.$$

Multiplying both members of equation (100) by $z(1-z)$ gives the hypergeometric equation:

$$(101) \qquad z(1-z)\frac{d^2u}{dz^2} + [c - (a+b+1)z]\frac{du}{dz} - abu = 0.$$

We now assume a solution of equation (101) in a neighborhood of $z = 0$ to be of the form:

$$u(z) = \sum_{r=0}^{\infty} c_r z^{\lambda+r} \; ; \quad c_o \neq 0, \text{ or}$$

$$(102) \qquad u(z) = c_o z^{\lambda} + c_1 z^{\lambda+1} + \cdots + c_{r-1} z^{\lambda+r-1} + c_r z^{\lambda+r} + \cdots .$$

Differentiating equation (102) with respect to z gives:

$$(103) \qquad u'(z) = c_o \lambda z^{\lambda-1} + c_1(\lambda+1)z^{\lambda} + \cdots + c_{r-1}(\lambda+r-1)z^{\lambda+r-2} + c_r(\lambda+r)z^{\lambda+r-1} + \cdots .$$

Differentiating equation (103) with respect to z yields:

$$(104) \qquad u''(z) = c_o \lambda(\lambda-1)z^{\lambda-2} + c_1(\lambda+1)(\lambda)z^{\lambda-1} + \cdots + c_{r-1}(\lambda+r-1)(\lambda+r-2)z^{\lambda+r-3} + c_r(\lambda+r)(\lambda+r-1)z^{\lambda+r-2} + \cdots .$$

Substituting equations (102) - 104) into

equation (101) gives us:

$$(105)\quad z(1-z)[c_o\lambda(\lambda-1)z^{\lambda-2} + c_1(\lambda+1)(\lambda)z^{\lambda-1} + \cdots + c_{r-1}(\lambda+r-1)(\lambda+r-2)z^{\lambda+r-3} + c_r(\lambda+r)(\lambda+r-1)z^{\lambda+r-2} + \cdots] + [c - (a+b+1)z][c_o\lambda z^{\lambda-1} + c_1(\lambda+1)z^{\lambda} + \cdots + c_{r-1}(\lambda+r-1)z^{\lambda+r-2} + c_r(\lambda+r)z^{\lambda+r-1} + \cdots] - ab[c_o z^{\lambda} + c_1 z^{\lambda+1} + \cdots + c_{r-1}z^{\lambda+r-1} + c_r z^{\lambda+r} + \cdots] = 0.$$

We now get the indicial equation by equating the coefficient of $z^{\lambda-1}$ to zero:

$$(106)\quad c_o\lambda(\lambda-1) + c_o\lambda c = 0.$$

Solving equation (106) yields:

$$(107)\quad \lambda = 0,\ 1-c.$$

To get the recurrence equation we equate the coefficient of $z^{\lambda+r-1}$ to zero thus:

$$(108)\quad c_r(\lambda+r)(\lambda+r-1) - c_{r-1}(\lambda+r-1)(\lambda+r-2) + c_r(\lambda+r)(c) - (a+b+1)c_{r-1}(\lambda+r-1) - abc_{r-1} = 0.$$

Solving equation (108) for $c_r$ gives the desired recurrence equation as follows:

$$(109)\quad c_r = \frac{[(\lambda+r-1)(\lambda+r-2) + (a+b+1)(\lambda+r-1) + ab]c_{r-1}}{(\lambda+r)(\lambda+r-1) + c(\lambda+r)}.$$

Equation (109) simplifies to:

$$(110)\quad c_r = \frac{(\lambda+r-1+a)(\lambda+r-1+b)c_{r-1}}{(\lambda+r)(\lambda+r-1+c)}.$$

In equation (112) we let $\lambda = 0$ and set $r = 1$, thus obtaining:

$$(111) \qquad c_1 = \frac{a \cdot b}{1 \cdot c} c_0 .$$

Let $r = 2$ in equation (110) to get

$c_2 = \frac{(a+1)(b+1)}{2(c+1)} c_1$ , and substituting $c_1$ from equation (111) yields:

$$(112) \qquad c_2 = \frac{(a)(a+1)(b)(b+1)}{1 \cdot 2\ (c)(c+1)} c_0 .$$

We now generalize by letting $r = n$ in equation (110), and in conjunction with the formation of equation (112), we deduce the following equation:

$$(113) \qquad c_n = \frac{(a)(a+1)\cdots(a+n-1)(b)(b+1)\cdots(b+n-1)c_0}{n!\ (c)(c+1)\cdots(c+n-1)} .$$

Substituting equation (113) into equation (102) gives a general solution of equation (101) in a neighborhood of $z = 0$ whenever $\lambda = 0$.

This solution is

$$(114) \qquad u_1(z) = K_1[1 + \sum_{n=1}^{\infty} \frac{(a)(a+1)\cdots(a+n-1)(b)(b+1)\cdots(b+n-1)}{n!\ (c)(c+1)\cdots(c+n-1)} z^n],$$

where $K_1$ is an arbitrary constant of integration.

Similar to the above analysis, we note that when $\lambda = 1-c$, equation (102) becomes:

$$(115) \qquad u_2(z) = K_2 z^{1-c}[1 + \sum_{n=1}^{\infty} \frac{(a-c+1)\cdots(a-c+n)(b-c+1)\cdots(b-c+n)}{n!\ (2-c)(3-c)\cdots(n+1-c)} z^n].$$

A general solution of equation (101) in a neighborhood of $z = 0$ is a linear combination of equations (114) and (115), thus:

$$(116)\quad u(z) = \{K_1[1 + \sum_{n=1}^{\infty} \frac{a(a+1)\cdots(a+n-1)(b)(b+1)\cdots(b+n-1)}{n!\ (c)(c+1)\cdots(c+n-1)} z^n]$$

$$+ K_2 z^{1-c}[1 + \sum_{n=1}^{\infty} \frac{(a-c+1)\cdots(a-c+n)(b-c+1)\cdots(b-c+n)}{n!\ (2-c)(3-c)\cdots(n+1-c)} z^n]\}.$$

We will now investigate the conditions under which the series given by equation (114) converges.

Let

$$(117)\quad S_1 = 1 + \sum_{n=1}^{\infty} \frac{(a)(a+1)\cdots(a+n-1)(b)(b+1)\cdots(b+n-1)}{n!\ (c)(c+1)\cdots(c+n-1)} z^n,$$

and let $v_n$ represent the n-th term in series (117). Then by the ratio test:

$$\left|\frac{v_{n+1}}{v_n}\right| = \left|\frac{(a+n-1)(b+n-1)}{n(c+n-1)} z\right|, \text{ or}$$

$$(118)\quad \left|\frac{v_{n+1}}{v_n}\right| = \left|\frac{(1 + \frac{a-1}{n})(1 + \frac{b-1}{n})}{1 + \frac{c-1}{n}}\right| \cdot |z|.$$

Now from equation (118),

$$(119)\quad \lim_{n \to \infty} \left|\frac{v_{n+1}}{v_n}\right| = |z|.$$ Hence the series given by equation (117) is absolutely and uniformly convergent if $|z| < 1$.

We will now establish the conditions under which series (117) is convergent if $|z| = 1$.

Let $|z| = 1$, then equation (118) becomes:

$$\left|\frac{v_{n+1}}{v_n}\right| = \left|[1 + \frac{a-1}{n}][1 + \frac{b-1}{n}][1 - \frac{c-1}{n} + O(1/n^2)]\right|. \tag{120}$$

In deriving equation (120), we note that

$$\left|\frac{1}{1 + \frac{c-1}{n}}\right| = \left|1 - \frac{c-1}{n} + O(1/n^2)\right|.$$

Equation (120) can be written:

$$\left|\frac{v_{n+1}}{v_n}\right| = \left|1 + \frac{a+b-2}{n}\right| \cdot \left|1 - \frac{c-1}{n} + O(1/n^2)\right|$$

$$= \left|1 - \frac{c-1}{n} + \frac{a+b-2}{n} + O(1/n^2)\right|.$$

$$\left|\frac{v_{n+1}}{v_n}\right| = \left|1 + \frac{a+b-c-1}{n} + O(1/n^2)\right|. \tag{121}$$

Now writing out the numbers a, b, and c, we get

$a = x_1 + iy_1$, $b = x_2 + iy_2$, and $c = x_3 + iy_3$, where $i^2 = -1$.

Substituting into equation (121) we get:

$$\left|\frac{v_{n+1}}{v_n}\right| = \left|1 + \frac{x_1+x_2-x_3-1}{n} + \frac{i(y_1+y_2-y_3)}{n} + O(1/n^2)\right|$$

$$= \{[1 + (x_1+x_2-x_3-1)/n]^2 + [(y_1+y_2-y_3)/n]^2 + O(1/n^2)\}^{1/2}.$$

$$\left|\frac{v_{n+1}}{v_n}\right| = \{[1 + (x_1+x_2-x_3-1)/n]^2 + O(1/n^2)\}^{1/2}. \tag{122}$$

Expanding equation (122) in a binomial expansion gives the following expression:

$$(123) \qquad \left|\frac{v_{n+1}}{v_n}\right| = 1 + (x_1+x_2-x_3-1)/n + O(1/n^2).$$

Now $S_1$ as given by equation (117) will be absolutely convergent for $|z| = 1$ whenever $x_1+x_2-x_3-1 < 1$, that is whenever $x_1+x_2-x_3 < 0$.

Expressed in a slightly different way, series (117) will be absolutely convergent when $|z| = 1$ if Re $(a+b-c) < 0$. [14, p. 24].

From the analysis of the convergence of equation (117), it follows that series (117) defines a function which is analytic when $|z| < 1$. This function is the hypergeometric function and is defined as follows:

$$(124) \qquad F(a,b;c;z) = 1 + \sum_{n=1}^{\infty} \frac{(a)(a+1)\cdots(a+n-1)(b)(b+1)\cdots(b+n-1)}{n!\,(c)(c+1)\cdots(c+n-1)} z^n.$$

Comparing equation (116) with equation (124) we see that the solution of equation (101) may be written in terms of the hypergeometric function. The result is:

$$(125) \qquad u(z) = K_1 F(a,b;c;z) + K_2 z^{1-c} F(a-c+1,\ b-c+1;\ 2-c;\ z).$$

This is the solution of equation (101) in a neighborhood of $z = 0$, expressed in terms of the hypergeometric function.

Solutions at the other regular singular points, unity and infinity, may be derived in a manner similar to the foregoing.

Some important functions which can be expressed in terms of hypergeometric functions are as follows:

(126) $(1+z)^n = F(-n,b;b;-z)$,

(127) $\ln(1+z) = zF(1,1;2;-z)$, and

(128) $e^z = \lim_{b \to \infty} F(1,b;1;z/b)$.

This has been a brief introduction to hypergeometric functions. However, they will be reviewed and used extensively in Chapter VIII when we solve certain differential equations with non-confluent regular singular points.

With the introductory material of Chapter I at our disposal, we are now ready to survey a linear ordinary second order differential equation with five regular singular points, and certain special cases. This will be accomplished in Chapter II.

## CHAPTER II

## DERIVATION OF A DIFFERENTIAL EQUATION WITH FIVE REGULAR SINGULAR POINTS, HAVING 1/2 AS THE EXPONENT DIFFERENCE AT EACH SINGULARITY

Subject to the restrictions outlined in Chapter I, we will now derive the most general differential equation of second order which has every point, except $z = a_r$, $r = 1,2,3,4$, and infinity, as an ordinary point. These five points are regular singular points with respective exponents $\alpha_r, \beta_r$, $r = 1,2,3,4$, and the exponents at infinity are $\gamma_1$, $\gamma_2$.

We will show that the above differential equation can be written thus:

$$(1)\quad \frac{d^2u}{dz^2} + \left[\sum_{r=1}^{4} \frac{1-\alpha_r-\beta_r}{z-a_r}\right]\frac{du}{dz} + \left[\sum_{r=1}^{4} \frac{\alpha_r\beta_r}{(z-a_r)^2} + \frac{Az^2+2Bz+C}{\prod_{r=1}^{4}(z-a_r)}\right]u = 0.$$

We begin by writing the differential equation thus:

$$(2)\quad \frac{d^2u}{dz^2} + p(z)\,\frac{du}{dz} + q(z)\,u = 0, \text{ where}$$

$$(3)\quad p(z) = \frac{\sum_{i=0}^{3} E_i z^i}{\prod_{r=1}^{4}(z-a_r)}, \text{ and}$$

$$(4)\quad q(z) = \frac{\sum_{j=0}^{6} F_j z^j}{\prod_{r=1}^{4}(z-a_r)^2}.$$

$E_i$, $i = 0,1,2,3$, and $F_j$, $j = 0,1,\cdots,6$ are finite constants.

We will now confirm that $\sum_{i=0}^{3} E_i z^i$ and $\sum_{j=0}^{6} F_j z^j$ were correctly chosen.

From equation (3) we note

(5) $p(z) = E_3 z^{-1} + O(z^{-2})$. Now let $w = 1/z$, and equation (5) becomes

(6) $p(1/w) = E_3 w + O(w^2)$, and thus

$\frac{1}{w} p(1/w) = E_3 + O(w)$ which, by analysis in Chapter I, is analytic at $w = 0$ for finite $E_3$. This implies by inversion that $zp(z)$ is analytic at infinity, hence infinity is a regular singular point, and $\sum_{i=0}^{3} E_i z^i$ was chosen correctly.

Similarly, $\sum_{j=0}^{6} F_j z^j$ was chosen correctly.

We now factor the right member of equation (3) into partial fractions thus:

(7) $$p(z) = \frac{\sum_{i=0}^{3} E_i z^i}{\prod_{r=1}^{4} (z-a_r)} = \sum_{r=1}^{4} \frac{G_r}{z-a_r}$$, and similarly

(8) $$q(z) = \frac{\sum_{j=0}^{6} F_j z^j}{\prod_{r=1}^{4} (z-a_r)^2} = \sum_{r=1}^{4} \frac{H_r}{(z-a_r)^2} + \sum_{r=1}^{4} \frac{J_r}{z-a_r}$$, where

$G_r$, $H_r$, and $J_r$ are constants for $r = 1,2,3,4$.

On the basis of equations (7) and (8), equation (2) becomes:

$$(9)\quad \frac{d^2u}{dz^2} + \left[\sum_{r=1}^{4} \frac{G_r}{z-a_r}\right]\frac{du}{dz} + \left[\sum_{r=1}^{4} \frac{H_r}{(z-a_r)^2} + \sum_{r=1}^{4} \frac{J_r}{z-a_r}\right]u = 0.$$

We note that $\sum_{r=1}^{4} \frac{H_r}{(z-a_r)^2}$ must be of the form

$$\frac{\sum_{i=0}^{6} A_i z^i}{\prod_{r=1}^{4}(z-a_r)^2}, \text{ and } \sum_{r=1}^{4} \frac{J_r}{z-a_r} \text{ must be of the form}$$

$$\frac{\sum_{i=0}^{2} B_i z^i}{\prod_{r=1}^{4}(z-a_r)} \text{ in order for infinity to be a regular}$$

singular point of equation (2).

We now assume a solution of equation (9) in a neighborhood of $z = a_1$ to be:

$$(10)\quad u(z) = c_o(z-a_1)^{\lambda} + \sum_{n=1}^{\infty} c_n(z-a_1)^{\lambda+n}; \text{ where } c_o \neq 0.$$

Differentiating equation (10) with respect to z, we get:

$$(11)\quad u'(z) = c_o\lambda(z-a_1)^{\lambda-1} + \sum_{n=1}^{\infty} c_n(\lambda+n)(z-a_1)^{\lambda+n-1}, \text{ and}$$

differentiating equation (11) with respect to z, we get:

$$(12)\quad u''(z) = c_o\lambda(\lambda-1)(z-a_1)^{\lambda-2} + \sum_{n=1}^{\infty} c_n(\lambda+n)(\lambda+n-1)(z-a_1)^{\lambda+n-2}.$$

Substitution of equations (10), (11), and (12) into equation (9) yields:

$$(13)\quad [c_0\lambda(\lambda-1)(z-a_1)^{\lambda-2} + \sum_{n=1}^{\infty} c_n(\lambda+n)(\lambda+n-1)(z-a_1)^{\lambda+n-2}]$$
$$+ \left[\sum_{r=1}^{4} \frac{G_r}{z-a_r}\right]\cdot[c_0\lambda(z-a_1)^{\lambda-1} + \sum_{n=1}^{\infty} c_n(\lambda+n)(z-a_1)^{\lambda+n-1}]$$
$$+ \left[\sum_{r=1}^{4} \frac{H_r}{(z-a_r)^2} + \sum_{r=1}^{4} \frac{J_r}{z-a_r}\right]\cdot[c_0(z-a_1)^{\lambda} + \sum_{n=1}^{\infty} c_n(z-a_1)^{\lambda+n}] = 0.$$

We are expanding in ascending powers of $(z-a_1)$; consequently to obtain the indicial equation, we equate the coefficient of $(z-a_1)^{\lambda-2}$ to zero, thus:

$$(14)\quad c_0\lambda(\lambda-1) + c_0\lambda g(G_1,G_2,G_3,G_4) + c_0 h(H_1,H_2,H_3,H_4) = 0,$$

where $g(G_1,G_2,G_3,G_4)$ and $h(H_1,H_2,H_3,H_4)$ are constants which we now determine.

From equation (13), let

$$(15)\quad S_4 = c_0\lambda(z-a_1)^{\lambda-1} \sum_{r=1}^{4} \frac{G_r}{z-a_r},$$

$$(16)\quad S_5 = \left[\sum_{r=1}^{4} \frac{G_r}{z-a_r}\right]\cdot\left[\sum_{n=1}^{\infty} c_n(\lambda+n)(z-a_1)^{\lambda+n-1}\right],$$

$$(17)\quad S_6 = c_0(z-a_1)^{\lambda} \sum_{r=1}^{4} \frac{H_r}{(z-a_r)^2},$$

$$(18)\quad S_7 = \left[\sum_{r=1}^{4} \frac{H_r}{(z-a_r)^2}\right] \cdot \left[\sum_{n=1}^{\infty} c_n (z-a_1)^{\lambda+n}\right],$$

$$(19)\quad S_8 = c_o (z-a_1)^{\lambda} \sum_{r=1}^{4} \frac{J_r}{z-a_r}, \text{ and}$$

$$(20)\quad S_9 = \left[\sum_{r=1}^{4} \frac{J_r}{z-a_r}\right] \cdot \left[\sum_{n=1}^{\infty} c_n (z-a_1)^{\lambda+n}\right].$$

Evidently, equation (16), and equations (18) - (20) contribute nothing to equation (14), hence we examine equations (15) and (17).

We may write equation (15) as follows:

$$S_4 = c_o\lambda(z-a_1)^{\lambda-1} \sum_{r=2}^{4} \frac{G_r}{(z-a_1)+(a_1-a_r)} + c_o G_1 \lambda (z-a_1)^{\lambda-2}$$

$$= c_o\lambda(z-a_1)^{\lambda-1} \sum_{r=2}^{4} \frac{G_r}{a_1-a_r} \cdot \frac{1}{1+\dfrac{z-a_1}{a_1-a_r}} + c_o G_1 \lambda (z-a_1)^{\lambda-2}.$$

$$(21)\quad S_4 = c_o\lambda(z-a_1)^{\lambda-1} \sum_{r=2}^{4} \frac{G_r}{a_1-a_r} \cdot \sum_{j=0}^{\infty} (-1)^j \left(\frac{z-a_1}{a_1-a_r}\right)^j + c_o G_1 \lambda (z-a_1)^{\lambda-2}.$$

We observe from equation (21) that the only coefficient of $(z-a_1)^{\lambda-2}$ is $c_o G_1 \lambda$, hence from equation (14),

(22) $g(G_1,G_2,G_3,G_4) = G_1$.

Similarly we may write equation (17) as follows:

(23) $$S_6 = c_o(z-a_1)^{\lambda} \sum_{r=2}^{4} \frac{H_r}{(a_1-a_r)^2}\left[\sum_{j=0}^{\infty}(-1)^j \left(\frac{z-a_1}{a_1-a_r}\right)^j\right]^2$$

$+ c_oH_1(z-a_1)^{\lambda-2}$, and also note from equation (14):

(24) $h(H_1,H_2,H_3,H_4) = H_1$. Substitution of equations (22) and (24) into equation (14) and division of the resulting expression by $c_o$ yields:

(25) $\lambda(\lambda-1) + \lambda G_1 + H_1 = 0$, or in alternate form:

(26) $\lambda^2 + \lambda(G_1-1) + H_1 = 0$.

Now the roots of equation (26) are the exponents at $z = a_1$, hence we have

$\alpha_1 + \beta_1 = 1-G_1$, or

(27) $G_1 = 1-\alpha_1-\beta_1$. Similarly,

(28) $H_1 = \alpha_1\beta_1$.

In general if we expand about $z = a_r$, then

(29) $G_r = 1-\alpha_r-\beta_r$; $r = 1,2,3,4$, and

(30) $H_r = \alpha_r\beta_r$.

Substituting equations (29) and (30) into equation (9) yields:

$$(31)\quad \frac{d^2u}{dz^2} + \left[\sum_{r=1}^{4} \frac{1-\alpha_r-\beta_r}{z-a_r}\right]\frac{du}{dz} + \left[\sum_{r=1}^{4} \frac{\alpha_r\beta_r}{(z-a_r)^2} + \sum_{r=1}^{4} \frac{J_r}{z-a_r}\right] u = 0.$$

We now recall that

$$(32)\quad \sum_{r=1}^{4} \frac{J_r}{z-a_r} = \frac{\sum_{i=0}^{2} B_i z^i}{\prod_{r=1}^{4}(z-a_r)}.$$

In equation (32) we arbitrarily let

$B_0 = C$, $B_1 = 2B$, and $B_2 = A$, whence

$$(33)\quad \sum_{r=1}^{4} \frac{J_r}{z-a_r} = \frac{Az^2+2Bz + C}{\prod_{r=1}^{4}(z-a_r)}.$$

Substituting equation (33) into equation (31) yields the desired form of the differential equation. That is,

$$(34)\quad \frac{d^2u}{dz^2} + \left[\sum_{r=1}^{4} \frac{1-\alpha_r-\beta_r}{z-a_r}\right]\frac{du}{dz} + \left[\sum_{r=1}^{4} \frac{\alpha_r\beta_r}{(z-a_r)^2} + \frac{Az^2 + 2Bz + C}{\prod_{r=1}^{4}(z-a_r)}\right] u = 0.$$

We pause in our derivations to compute A in equation (34) in terms of $\alpha_r$ after applying the restriction

$$(35)\quad \beta_r - \alpha_r = 1/2.$$ Following this, we will show that

the exponents at infinity may be expressed in terms of $\alpha_r$.

Initially we assume a solution of equation (34) in a neighborhood of infinity to be:

$$(36) \qquad u(z) = c_o z^{-\lambda} + \sum_{n=1}^{\infty} c_{-n} z^{-\lambda-n}; \quad c_o \neq 0.$$

Differentiating equation (36) with respect to z yields:

$$(37) \qquad u'(z) = c_o(-\lambda)z^{-\lambda-1} + \sum_{n=1}^{\infty} c_{-n}(-\lambda-n)z^{-\lambda-n-1}, \text{ and}$$

differentiating equation (37) with respect to z yields:

$$(38) \qquad u''(z) = c_o(-\lambda)(-\lambda-1)z^{-\lambda-2} + \sum_{n=1}^{\infty} c_{-n}(-\lambda-n)(-\lambda-n-1)z^{-\lambda-n-2}.$$

We now substitute equations (36) - (38) into equation (34) to get:

$$(39) \qquad c_o(-\lambda)(-\lambda-1)z^{-\lambda-2} + \sum_{n=1}^{\infty} c_{-n}(-\lambda-n)(-\lambda-n-1)z^{-\lambda-n-2}$$

$$+ \left[\sum_{r=1}^{4} \frac{1-\alpha_r-\beta_r}{z-a_r}\right] \cdot \left[c_o(-\lambda)z^{-\lambda-1} + \sum_{n=1}^{\infty} c_{-n}(-\lambda-n)z^{-\lambda-n-1}\right]$$

$$+ \left[\sum_{r=1}^{4} \frac{\alpha_r\beta_r}{(z-a_r)^2} + \frac{Az^2 + 2Bz + C}{\prod_{r=1}^{4}(z-a_r)}\right] \cdot \left[c_o z^{-\lambda} + \sum_{n=1}^{\infty} c_{-n} z^{-\lambda-n}\right] = 0.$$

Inasmuch as we are expanding in a neighborhood of infinity, equation (36) is an expansion in descending powers of z. Thus the indicial equation is found from equation (39)

by equating to zero the coefficient of $z^{-\lambda-2}$, where $(-\lambda-2)$ is greater than or equal to all other exponents on z.

From equation (39), we let

$$S_1 = c_o(-\lambda)(-\lambda-1)z^{-\lambda-2}, \tag{40}$$

$$S_2 = c_o(-\lambda)z^{-\lambda-1} \sum_{r=1}^{4} \frac{1-\alpha_r-\beta_r}{z-a_r}, \tag{41}$$

$$S_3 = c_o z^{-\lambda} \sum_{r=1}^{4} \frac{\alpha_r\beta_r}{(z-a_r)^2}, \tag{42}$$

$$S_4 = c_o z^{-\lambda} \frac{(Az^2 + 2Bz + C)}{\prod_{r=1}^{4} (z-a_r)}, \tag{43}$$

$$S_5 = \sum_{n=1}^{\infty} c_{-n}(-\lambda-n)(-\lambda-n-1)z^{-\lambda-n-2}, \tag{44}$$

$$S_6 = \left[\sum_{r=1}^{4} \frac{1-\alpha_r-\beta_r}{z-a_r}\right] \cdot \left[\sum_{n=1}^{\infty} c_{-n}(-\lambda-n)z^{-\lambda-n-1}\right], \tag{45}$$

$$S_7 = \sum_{r=1}^{4} \frac{\alpha_r\beta_r}{(z-a_r)^2} \sum_{n=1}^{\infty} c_{-n}z^{-\lambda-n} \quad \text{and} \tag{46}$$

$$S_8 = \frac{Az^2 + 2Bz + C}{\prod_{r=1}^{4} (z-a_r)} \sum_{n=1}^{\infty} c_{-n}z^{-\lambda-n}. \tag{47}$$

By inspection we note that equations (44) - (48) contribute nothing to the indicial equation. We now examine equations (40) - (43) more closely.

The coefficient of $z^{-\lambda-2}$ equated to zero is:

$$c_o(-\lambda)(-\lambda-1) + c_o(-\lambda)T_2 + c_oT_3 + c_oT_4 = 0, \tag{48}$$

where $T_j$ are constants which can be determined from the expression for $S_j$, $(j = 2,3,4)$.

From equation (41)

$$S_2 = c_o(-\lambda)z^{-\lambda-1} \sum_{r=1}^{4} (1-\alpha_r-\beta_r)/(z-a_r)$$

$$= c_o(-\lambda)z^{-\lambda-1} \sum_{r=1}^{4} [(1-\alpha_r-\beta_r)/z] \frac{1}{1 - a_r/z}$$

$$= c_o(-\lambda)z^{-\lambda-1} \sum_{r=1}^{4} [(1-\alpha_r-\beta_r)/z] \sum_{j=0}^{\infty} (-1)^j(a_r/z)^j, \text{ or}$$

$$S_2 = c_o(-\lambda)z^{-\lambda-2} \sum_{r=1}^{4} (1-\alpha_r-\beta_r) \sum_{j=0}^{\infty} (-1)^j(a_r/z)^j. \tag{49}$$

Comparing equations (48) and (49), we see that

$$T_2 = \sum_{r=1}^{4} (1-\alpha_r-\beta_r). \tag{50}$$

Similar to the above analysis, we may write equation (42) thus:

$$S_3 = c_o z^{-\lambda} \sum_{r=1}^{4} \frac{\alpha_r \beta_r}{z^2} \cdot \frac{1}{(1 - a_r/z)^2} \quad , \text{ or}$$

$$(51) \qquad S_3 = c_o z^{-\lambda-2} \sum_{r=1}^{4} \alpha_r \beta_r \left[ \sum_{j=0}^{\infty} (-1)^j (a_r/z)^j \right]^2 .$$

Comparing equations (51) and (48) shows that

$$(52) \qquad T_3 = \sum_{r=1}^{4} \alpha_r \beta_r .$$

We may write equation (43) as follows:

$$S_4 = c_o z^{-\lambda} \frac{(A/z^2 + 2B/z^3 + C/z^4)}{\prod_{r=1}^{4} (1 - a_r/z)}$$

$$= c_o z^{-\lambda} (A/z^2 + 2B/z^3 + C/z^4) \prod_{r=1}^{4} \left[ \sum_{j=0}^{\infty} (-1)^j (a_r/z)^j \right]$$

$$= c_o z^{-\lambda} [A/z^2 + O(z^{-3})] \quad , \text{ or finally}$$

$$(53) \qquad S_4 = A c_o z^{-\lambda-2} + c_o z^{-\lambda} [O(z^{-3})].$$

Comparing equations (53) and (48) shows that

$$(54) \qquad T_4 = A.$$

Substituting equations (50), (52) and (54) into equation (48), and remembering that $c_o \neq 0$ yields the following expression for the indicial equation:

$$(55) \qquad \lambda(\lambda+1) - \lambda \sum_{r=1}^{4} (1-\alpha_r-\beta_r) + \sum_{r=1}^{4} \alpha_r\beta_r + A = 0.$$

We now apply the restriction given by equation (35) to equation (55), and recall that infinity is an elementary singularity to get: [5, p. 495]

$$\lambda(\lambda+1) - \lambda \sum_{r=1}^{4} (1-\alpha_r-\alpha_r-1/2) + \sum_{r=1}^{4} \alpha_r(\alpha_r+1/2) + A = 0,$$

or upon rearrangement,

$$(56) \qquad \lambda^2 + \lambda \left[-1 + 2 \sum_{r=1}^{4} \alpha_r\right] + \sum_{r=1}^{4} \alpha_r(\alpha_r + 1/2) + A = 0.$$

Let the roots of equation (56) be $\lambda_1$ and $\lambda_2$, then

$$(57) \qquad \lambda_2 - \lambda_1 = \left\{\left[2 \sum_{r=1}^{4} \alpha_r - 1\right]^2 - 4\left[\sum_{r=1}^{4} \alpha_r(\alpha_r + 1/2) + A\right]\right\}^{1/2} = \tfrac{1}{2}.$$

We square both members of equation (57) and simplify to get:

$$(58) \qquad A = \left[\sum_{r=1}^{4} \alpha_r\right]^2 - \sum_{r=1}^{4} \alpha_r^2 - 3/2 \sum_{r=1}^{4} \alpha_r + 3/16.$$

Equation (58) expresses A, in equation (34), as a function of $\alpha_r$ only. We now substitute equation (58) into equation (56) and simplify to get:

(59) $$\lambda^2 + \lambda[2 \sum_{r=1}^{4} \alpha_r - 1] + \left[\sum_{r=1}^{4} \alpha_r\right]^2 - \sum_{r=1}^{4} \alpha_r + 3/16 = 0.$$

Solving equation (59) for the exponents at infinity we get:

$$\lambda = \frac{[1 - 2 \sum_{r=1}^{4} \alpha_r] \pm R}{2}, \text{ that is,}$$

(60) $$\lambda_2 \equiv \gamma_2 = \frac{[1 - 2 \sum_{r=1}^{4} \alpha_r] + R}{2}, \text{ and}$$

(61) $$\lambda_1 \equiv \gamma_1 = \frac{[1 - 2 \sum_{r=1}^{4} \alpha_r] - R}{2}, \text{ where } \gamma_1 \text{ and } \gamma_2 \text{ are the}$$

exponents at infinity, and

$$R^2 = [2\Sigma\alpha_r - 1]^2 - 4[(\Sigma\alpha_r)^2 - \Sigma\alpha_r + 3/16], \text{ or}$$

(62) $R^2 = 1/4$. Substituting the values of R given by equation (62) into equations (60) and (61) yields:

(63) $$\gamma_1 = 1/4 - \sum_{r=1}^{4} \alpha_r, \text{ and}$$

(64) $$\gamma_2 = 3/4 - \sum_{r=1}^{4} \alpha_r.$$

Equations (63) and (64) display the exponents at infinity as functions of $\alpha_r$ only. Further, we note that infinity is an elementary singular point since the exponent difference at infinity is 1/2. [5, p. 495 ].

We continue the derivation of certain differential equations of mathematical physics by first obtaining the generalized Lamé equation, [14, p. 204]. This is accomplished by applying restriction (35) to equation (34), thus:

$$(65)\quad \frac{d^2u}{dz^2} + \left[\sum_{r=1}^{4} \frac{1/2 - 2\alpha_r}{z-a_r}\right]\frac{du}{dz}$$

$$+ \left[\sum_{r=1}^{4} \frac{\alpha_r(\alpha_r + 1/2)}{(z-a_r)^2} + \frac{Az^2 + 2Bz + C}{\prod_{r=1}^{4}(z-a_r)}\right]u = 0,$$

where A is given by equation (58), and B and C are constants.

We will now specialize equation (65).

## Lamé's Equation

Equation (65) may be written as follows:

$$(66)\quad \prod_{r=1}^{3}(z-a_r)\frac{d^2u}{dz^2} + \left[\sum_{r=1}^{3} \frac{1/2 - 2\alpha_r}{z-a_r} + \frac{1/2 - \alpha_4}{z-a_4}\right]\prod_{r=1}^{3}(z-a_r)\frac{du}{dz}$$

$$+ \prod_{r=1}^{3}(z-a_r)\left[\sum_{r=1}^{3} \frac{\alpha_r(\alpha_r + 1/2)}{(z-a_r)^2} + \frac{\alpha_4(\alpha_4 + 1/2)}{z-a_4} + \frac{Az^2+2Bz+C}{z-a_4}\right]u = 0.$$

In equation (66), we set $\alpha_r = 0$, $(r = 1,2,3,4)$, let $a_4 \to \infty$, let $8B = n(n+1)a_4$, $C = ha_4/4$, $A = 3/16$, then divide both numbers by $\prod_{r=1}^{3}(z-a_r)$ to get:

$$(67)\quad \frac{d^2u}{dz^2} + \left[\sum_{r=1}^{3} \frac{1}{2(z-a_r)}\right]\frac{du}{dz}$$

$$+ \left[\frac{\lim_{a_4 \to \infty} \dfrac{3z^2 + z(n^2+n)a_4 + ha_4}{4(z-a_4)}}{\dfrac{3}{\prod_{r=1}} (z-a_r)}\right] u = 0.$$

Equation (67) immediately simplifies to

$$(68)\quad \frac{d^2u}{dz^2} + \left[\sum_{r=1}^{3} \frac{1}{2(z-a_r)}\right]\frac{du}{dz} - \left[\frac{n(n+1)z + h}{4\prod_{r=1}^{3}(z-a_r)}\right] u = 0.$$

We classify equation (68), in the notation of Ince [ 5, p. 497 ], as an equation of the [3,1,0] type.[2] Equation (68) is Lamé's equation.

## Mathieu's Equation

In order to derive Mathieu's equation, we consider equation (65) and let $\alpha_1 = \alpha_2 = 0$, $\beta_1 = \beta_2 = 1/2$, $a_1 = 0$, $a_2 = 1$, $a_3 \to \infty$, and $a_4 \to \infty$. From this information, and from equation (38), $A = 2\alpha_3\alpha_4 - 3\alpha_3/2 - 3\alpha_4/2 + 3/16$. We now choose $2B = -8qa_3a_4$, and $C = -(a - 16q)a_3a_4/4$.

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[2]The expression [a,b,c] characterizes an equation with the following properties:
a = the number of elementary singularities,
b = the number of non-elementary regular singularities, and
c = the number of essential singularities.

We substitute the preceding information into equation (65) to get:

$$(69)\quad \frac{d^2u}{dz^2} + \left[\frac{1}{2z} + \frac{1}{2(z-1)} + \frac{1/2 - 2\alpha_3}{(z-a_3)} + \frac{1/2 - 2\alpha_4}{(z-a_4)}\right]\frac{du}{dz}$$

$$+ \left[\frac{\alpha_3(\alpha_3 + 1/2)}{(z-a_3)} + \frac{\alpha_4(\alpha_4 + 1/2)}{(z-a_4)}\right.$$

$$\left. + \frac{(2\alpha_3\alpha_4 - \frac{3}{2}\alpha_3 - \frac{3}{2}\alpha_4 + \frac{3}{16})z^2 - (8qa_3a_4)z - (a-16q)a_3a_4)/4}{z(z-1)(z^2 - a_3z - a_4z + a_3a_4)}\right]u = 0.$$

We rearrange equation (69) slightly thus:

$$(70)\quad \frac{d^2u}{dz^2} + \left[\frac{1}{2z} + \frac{1}{2(z-1)} + \frac{1/2 - 2\alpha_3}{(z-a_3)} + \frac{1/2 - 2\alpha_4}{(z-a_4)}\right]\frac{du}{dz}$$

$$+ \left[\frac{\alpha_3(\alpha_3 + 1/2)}{(z-a_3)} + \frac{\alpha_4(\alpha_4 + 1/2)}{(z-a_4)}\right.$$

$$\left. + \frac{\frac{(2\alpha_3\alpha_4 - \frac{3}{2}\alpha_3 - \frac{3}{2}\alpha_4 + \frac{3}{16})z^2}{a_3a_4} - 8qz - \frac{(a-16q)}{4}}{z(z-1)(z^2/a_3a_4 - z/a_4 - z/a_3 + 1)}\right]u = 0.$$

In equation (70) we let $a_3 \to \infty$, $a_4 \to \infty$, hence $a_3a_4 \to \infty$, to get:

$$(71)\quad \frac{d^2u}{dz^2} + \left[\frac{1}{2z} + \frac{1}{2(z-1)}\right]\frac{du}{dz} - \left[\frac{a-16q+32qz}{4z(z-1)}\right]u = 0.$$

We digress at this point to investigate the method of removing $p(z)\frac{du}{dz}$ from equation (2).

We begin by making the variable transformation:

(72) $z = f(w)$, and recall that

(73) $\frac{du}{dz} = \frac{du}{dw} \cdot \frac{dw}{dz}$ , and

(74) $\frac{d^2u}{dz^2} = \frac{du}{dw} \cdot \frac{d^2w}{dz^2} + \frac{d^2u}{dw^2} \cdot \left[\frac{dw}{dz}\right]^2$ . Substitute equations (73) and (74) into equation (2) obtaining:

(75) $\frac{d^2u}{dw^2}\left[\frac{dw}{dz}\right]^2 + \left[\frac{d^2w}{dz^2} + p(z)\frac{dw}{dz}\right]\frac{du}{dw} + q(z)\ u = 0.$

We now wish to require that

(76) $\frac{d^2w}{dz^2} + p(z)\frac{dw}{dz} = 0.$ Then we let

(77) $\frac{dw}{dz} = v,$ hence equation (76) becomes:

(78) $\frac{dv}{dz} + p(z)v = 0.$ We separate variables in equation (78) and integrate to obtain:

(79) $v = c_1 e^{-\int p(z)dz}.$ Substitute equation (77) into equation (79) and integrate once more to obtain:

(80) $w = c_1 \int e^{-\int p(z)dz}\, dz + c_2$, where $c_1$ and $c_2$ are are arbitrary constants of integration.

Comparing equations (2) and (71) we see that

(81) $p(z) = \frac{1}{2z} + \frac{1}{2(z-1)}$. We now choose $c_2 = 0$, and substitute equation (81) into equation (80) to get:

$$w = c_1 \int e^{-\int dz/2z - \int dz/2(z-1)} dz$$

$$= c_1 \int e^{-\frac{1}{2}\ln z - \frac{1}{2}\ln(z-1)} dz.$$

$w = c_1 \int \frac{dz}{[z(z-1)]^{1/2}}$. This may be written as

(82) $w = c_1 \int \frac{dz}{[(z-1/2)^2 - 1/4]^{1/2}}$.

In equation (82), let $z - 1/2 = 1/2 \sec \theta$, then $dz = 1/2 \sec \theta \tan \theta \, d\theta$, and substitution into equation (82) gives us, after some simplification:

(83) $w = c_1 \int \sec \theta \, d\theta$, or finally,

(84) $w/c_1 = \ln (\sec \theta + \tan \theta)$. But $\sec \theta = 2z - 1$, hence $\tan \theta = 2(z^2 - z)^{1/2}$, hence equation (84) becomes:

$w/c_1 = \ln [2z-1 + 2(z^2-z)^{1/2}]$, or

(85) $e^{w/c_1} = (2z-1) + 2(z^2-z)^{1/2}$.

Solving equation (85) for z, we get

$z = \cosh^2 (w/2c_1)$, or

$z = \cos^2 (iw/2c_1)$. We now choose $c_1 = 1/2$ so that

(86) $z = \cos^2 w$.

Equation (86) is the desired transformation, which was outlined in general by equation (72).

For $z = \cos^2 w$, equation (73) becomes:

(87) $$\frac{du}{dz} = - \csc 2w \frac{du}{dw} .$$

From equation (86),

$w = \cos^{-1} \sqrt{z}$, whence

$$\frac{dw}{dz} = - \frac{1}{2\sqrt{z(1-z)}} , \text{ and}$$

$$\frac{d^2w}{dz^2} = \frac{1 - 2z}{4(z-z^2)^{3/2}} . \quad \text{But } z = \cos^2 w, \text{ hence}$$

(88) $$\frac{d^2w}{dz^2} = \frac{1 - 2\cos^2 w}{4(\cos^2 w - \cos^4 w)^{3/2}} = - \frac{2 \cos 2w}{\sin^3 2w} .$$

Substituting equations (87) and (88) into equation (74) yields:

(89) $$\frac{d^2u}{dz^2} = - \frac{2 \cos 2w}{\sin^3 2w} \cdot \frac{du}{dw} + (-1/\sin 2w)^2 \frac{d^2u}{dw^2} .$$

Substituting equations (87) and (89) into equation (71) we get:

$$(90)\qquad \csc^2 2w\,\frac{d^2u}{dw^2} - \frac{2\cos 2w}{\sin^3 2w}\cdot\frac{du}{dw}$$

$$- \left[\frac{1}{2\cos^2 w} + \frac{1}{2(\cos^2 w - 1)}\right]\csc 2w\,\frac{du}{dw}$$

$$- \left[\frac{a - 16q + 32q\cos^2 w}{4\cos^2 w(\cos^2 w - 1)}\right]u = 0.$$

Multiply both members of equation (90) by $\sin^2 2w$ to get:

$$(91)\qquad \frac{d^2u}{dw^2} + \left[\frac{-\sin 2w}{2\cos^2 w} + \frac{\sin 2w}{2\sin^2 w} - \frac{2\cos 2w}{\sin 2w}\right]\frac{du}{dw}$$

$$- \left[\frac{a - 16q + 32q\cos^2 w}{4(\cos^2 w)(\cos^2 w - 1)}\right](\sin^2 2w)u = 0.$$

In equation (91) we notice that

$$+\frac{\sin 2w}{2}\left[\frac{-\sin^2 w + \cos^2 w}{\sin^2 w\cos^2 w}\right] = \frac{\sin 2w\cos 2w}{2\,\frac{\sin^2 2w}{4}} = 2\cot 2w.$$

Also

$$4(\cos^2 w)(\cos^2 w - 1) = 4\sin^2 w\cos^2 w = \sin^2 2w.$$

Substitute these identities into equation (91) to get:

$$(92)\qquad \frac{d^2u}{dw^2} - (a - 16q + 32q\cos^2 w)u = 0.$$

Again we note that $\cos^2 w = 1/2 + 1/2\cos 2w$, and substitution into equation (92) yields:

$$\frac{d^2u}{dw^2} - [a - 16q + 32q(1/2 + 1/2 \cos 2w)]u = 0, \text{ or}$$

$$\frac{d^2u}{dw^2} - (a + 16q \cos 2w)u = 0. \tag{93}$$

Equation (93) is Mathieu's equation and is classified in Ince as being of the [2,0,1] type.

## Legendre's Equation

In order to derive Legendre's equation, we refer to equation (65), and let $a_1 = a_2 = 1$; $a_3 = a_4 = 0$, $\alpha_1 = \alpha_2 = \alpha_3 = 0$, and $\alpha_4 = 1/4$.

Substituting these values into equation (65) yields the following equation:

$$\frac{d^2u}{dz^2} + [1/2z + 1/(z-1)]\frac{du}{dz} + \left[\frac{3}{16z^2} + \frac{Az^2 + 2Bz + C}{z^2(z-1)^2}\right]u = 0. \tag{94}$$

We can write equation (94) as follows:

$$\frac{d^2u}{dz^2} + [1/2z + 1/(z-1)]\frac{du}{dz}$$

$$+ \frac{1}{4}\left[\frac{n(n+1)}{z} - \frac{m^2}{z-1}\right]\frac{u}{z(z-1)} = 0, \text{ compare the}$$

coefficients of u, and hence determine values for A, B, C in terms of m and n.

Thus

$$\frac{3}{16z^2} + \frac{Az^2 + 2Bz + C}{z^2(z-1)^2} \equiv \frac{1}{4}\left[\frac{n(n+1)}{z^2(z-1)} - \frac{m^2}{z(z-1)^2}\right].$$

Simplifying the above identity we get:

$$\frac{3}{16}(z-1)^2 + (Az^2 + 2Bz + C) \equiv \frac{1}{4}[n(n+1)(z-1) - m^2z],$$

$$\frac{3}{16}z^2 - \frac{3}{8}z + \frac{3}{16} + Az^2 + 2Bz + C$$

$$\equiv \frac{1}{4}[z(n^2+n-m^2) - n^2-n], \text{ or}$$

$$z^2(3/16 + A) + z(2B - 3/8) + (3/16 + C)$$

$$\equiv \frac{z(n^2+n-m^2)}{4} - \frac{n^2+n}{4}.$$

Equating coefficients of like powers of z we obtain the following set of equations:

(95) $A = -3/16$

$$B = \frac{3 + 2(n^2+n-m^2)}{16}$$

(97) $$C = \frac{-(4n^2+4n+3)}{16}$$

We now make the variable transformation

(98) $z = w^{-2}$, and deduce from equations (73) and (74) that

(99) $$\frac{du}{dz} = -\frac{1}{2}w^3 \frac{du}{dw}, \text{ and}$$

(100) $$\frac{d^2u}{dz^2} = \frac{1}{4}w^6 \frac{d^2u}{dw^2} + \frac{3}{4}w^5 \frac{du}{dw} .$$

We now substitute equations (95) - (100) into equation (94) to get:

(101) $$\left[\frac{1}{4}w^6 \frac{d^2u}{dw^2} + \frac{3}{4}w^5 \frac{du}{dw}\right] + \left[\frac{w^2}{2} + \frac{1}{w^{-2} - 1}\right]\left[-\frac{1}{2}w^3 \frac{du}{dw}\right]$$

$$+ \left[\frac{3w^4}{16} + \frac{-\frac{3}{16}w^{-4} + \frac{(3+2n^2+2n-2m^2)w^{-2}}{8} - \frac{4n^2+4n+3}{16}}{w^{-4}(w^{-2} - 1)^2}\right]u = 0.$$

We now simplify equation (101). The coefficient of $\frac{du}{dw}$ is:

$$\frac{3}{4}w^5 - \frac{w^5}{4} - \frac{w^5}{2(1-w^2)} = \frac{1}{2}w^5 - \frac{w^5}{2(1-w^2)} .$$

Let T represent the coefficient of u, then equation (101) becomes:

$$(102)\qquad \frac{1}{4}w^6\frac{d^2u}{dw^2} + w^5\left[\frac{1}{2} - \frac{1}{4(1-w^2)}\right]\frac{du}{dw} + Tu = 0,$$

where

$$(103)\qquad T = \frac{3w^4}{16} + \frac{-3w^{-4} + (6+4n^2+4n+4m^2)w^{-2} - 4n^2-4n-3}{16w^{-4}(w^{-2}-1)^2}$$

$$= \frac{3w^4}{16} + \frac{-3w^4 + (6+4n^2+4n+4m^2)w^6 - (4n^2+4n+3)w^8}{16(1-w^2)^2}$$

$$= [3w^4 - 6w^6 + 3w^8 - 3w^4 + 6w^6 + 4n^2w^6 + 4nw^6 + 4m^2w^6$$

$$- 4n^2w^8 - 4nw^8 - 3w^8]/[16(1-w^2)^2]$$

$$= \frac{n^2w^6 + nw^6 + m^2w^6 - n^2w^8 - nw^8}{4(1-w^2)^2}$$

$$= \frac{nw^2(n+1)(1-w^2)}{4(1-w^2)^2} + \frac{m^2w^6}{4(1-w^2)^2}\text{ , or finally,}$$

$$(104)\qquad T = \frac{w^6}{4}\left[\frac{n(n+1)}{1-w^2} + \frac{m^2}{(1-w^2)^2}\right].$$

Substitute equation (104) into equation (102) to get:

$$(105)\qquad \frac{1}{4}w^6\frac{d^2u}{dw^2} + w^5\left[\frac{1}{2} - \frac{1}{2(1-w^2)}\right]\frac{du}{dw}$$

$$+ \frac{w^6}{4}\left[\frac{n(n+1)}{1-w^2} + \frac{m^2}{(1-w^2)^2}\right]u = 0.$$

Multiplying both members of equation (105) by $\frac{4(1-w^2)}{w^6}$, we finally obtain Legendre's equation:

$$(106) \quad (1-w^2)\frac{d^2u}{dw^2} - 2w\frac{du}{dw} + [n(n+1) - m^2/(1-w^2)]u = 0.$$

This equation has been classified by Ince as an equation of the [1,2,0] class.

## Bessel's Equation

From equation (65), we let $\alpha_r = 0$, $r = 1,2,3,4$; $a_1 = a_2 = 0$; $a_3 = a_4$, and we let $a_3 \to \infty$. Substitution of this information into equation (65) yields the following equation:

$$(107) \quad \frac{d^2u}{dz^2} + \frac{1}{z}\cdot\frac{du}{dz} + \underset{a_3 \to \infty}{\text{Lim}} \left[\frac{Az^2 + 2Bz + C}{z^2(z-a_3)^2}\right]u = 0.$$

From equation (58), $A = 3/16$, and we arbitrarily choose $2B = \frac{1}{4}a_3^2$, and $C = -\frac{n^2a_3^2}{4}$.

Substitution of these values into equation (107) gives us:

$$(108)\qquad \frac{d^2u}{dz^2} + \frac{1}{z}\cdot\frac{du}{dz} + \lim_{a_3 \to \infty}\left[\frac{\frac{3}{16}z^2 + \frac{1}{4}a_3^2 z - \frac{n^2a_3^2}{4}}{z^2(z-a_3)^2}\right]u = 0.$$

We may rearrange equation (108) slightly as follows:

$$z^2\frac{d^2u}{dz^2} + z\frac{du}{dz} + \lim_{a_3 \to \infty}\left[\frac{\frac{3z^2}{16a_3^2} + \frac{z}{4} - \frac{n^2}{4}}{(z/a_3-1)^2}\right]u = 0,$$

and hence equation (108) finally becomes:

$$(109)\qquad z^2\frac{d^2u}{dz^2} + z\frac{du}{dz} + \frac{1}{4}(z-n^2)u = 0.$$

We now make the variable transformation:

(110) $z = w^2$. Then in accordance with this transformation, equations (73) and (74) yield:

$$(111)\qquad \frac{du}{dz} = \frac{1}{2w}\cdot\frac{du}{dw}, \text{ and}$$

$$(112)\qquad \frac{d^2u}{dz^2} = \frac{\frac{d^2u}{dw^2} - \frac{1}{w}\cdot\frac{du}{dw}}{4w^2}.$$

Substitute equations (110) - (112) into equation (109) to obtain:

$$(113)\qquad \frac{w^4}{4w^2}\left[\frac{d^2u}{dw^2} - \frac{1}{w}\frac{du}{dw}\right] + \frac{w^2}{2w}\cdot\frac{du}{dw} + \frac{1}{4}(w^2-n^2)u = 0.$$

Simplifying equation (113) yields:

$$w^2\frac{d^2u}{dw^2} - (w-2w)\frac{du}{dw} + (w^2-n^2)u = 0, \text{ or finally:}$$

$$(114)\qquad w^2\frac{d^2u}{dw^2} + w\frac{du}{dw} + (w^2-n^2)u = 0.$$

Equation (114) is Bessel's equation and is classified by Ince as an equation of the [0,1,1] class.

## Weber's Equation

We begin the derivation of Weber's equation by considering equation (65) where we let $\alpha_r = 0$; $r = 1,2,3,4$; $a_1 = 0$; and $a_2 = a_3 = a_4$; and $a_2 \to \infty$. From equation (58) we note that $A = 3/16$. We choose $B = a_2^3/32$, and $C = -(n+1/2)a_2^3/4$.

Substitution of the above information into equation (65) gives us the following equation:

$$(115)\qquad \frac{d^2u}{dz^2} + \frac{1}{2z}\cdot\frac{du}{dz} + \lim_{a_2 \to \infty}\left[\frac{3z^2/16 + a_2^3 z/16 - (n+\frac{1}{2})a_2^3/4}{z(z-a_2)^3}\right]u = 0.$$

Equation (115) may be rearranged slightly as follows:

$$(116)\qquad \frac{d^2u}{dz^2} + \frac{1}{2z}\cdot\frac{du}{dz} + \lim_{a_2 \to \infty}\left[\frac{\frac{3}{16}\cdot\frac{z^2}{a_2^3} + z/16 - (n+\frac{1}{2})/4}{z(z/a_2^3 - 1)^3}\right]u = 0,$$

which becomes upon letting $a_2 \to \infty$,

$$(117)\qquad \frac{d^2u}{dz^2} + \frac{1}{2z}\cdot\frac{du}{dz} + \left[\frac{(n+1/2)/4 - z/16}{z}\right]u = 0, \text{ or}$$

$$(118)\qquad z\frac{d^2u}{dz^2} + \frac{1}{2}\frac{du}{dz} + [(n+1/2)/4 - z/16]u = 0.$$

We now make the variable transformation:

$$(119)\qquad z = w^2.$$

In order to verify that $z = w^2$ is the desired transformation which will remove the first derivative term from equation (117), we refer to equation (80), in which $p(z) = \frac{1}{2z}$. We then obtain from equation (80):

$$w = c_1 \int e^{-\int dz/2z}\, dz + c_2$$

$$= c_1 \int e^{-(\ln z)/2}\, dz + c_2$$

$$= c_1 \int z^{-1/2}\, dz + c_2, \text{ or}$$

$w = 2c_1(z^{1/2}) + c_2$, where, as before, $c_1$ and $c_2$ are arbitrary constants of integration.

Choose $c_2 = 0$, and $c_1 = 1/2$, then $w = z^{1/2}$, or $z = w^2$, as desired.

Utilizing equations (73) and (74) in accordance with equation (119) yields:

(120) $$\frac{du}{dz} = \frac{1}{2w} \cdot \frac{du}{dw}, \text{ and}$$

(121) $$\frac{d^2u}{dz^2} = \frac{1}{4w^2} \cdot \frac{d^2u}{dw^2} - \frac{1}{4w^3} \cdot \frac{du}{dw}.$$

We now substitute equations (119), (120), and (121) into equation (118) to get:

(122) $$w^2 \left[\frac{1}{4w^2} \cdot \frac{d^2u}{dw^2} - \frac{1}{4w^3} \cdot \frac{du}{dw}\right] + \frac{1}{2}\left[\frac{1}{2w} \cdot \frac{du}{dw}\right]$$

$$+ [(n + 1/2)/4 - w^2/16]u = 0.$$

Multiplying both members of equation (122) by 4 gives the following equation:

$$(123) \qquad \frac{d^2u}{dw^2} + [(n + 1/2) - w^2/4]u = 0.$$

Equation (123) is Weber's equation, and is an equation of the [1,0,1] class.

## Stokes' Equation

To derive Stokes' equation, we refer again to equation (65), let $\alpha_r = 0$, $(r = 1,2,3,4)$, and let $a_r \to \infty$, $(r = 1,2,3,4)$. We then get from equation (65):

$$(124) \qquad \frac{d^2u}{dz^2} + \operatorname*{Lim}_{a_1 \to \infty} \left[\frac{Az^2 + 2Bz + C}{(z-a_1)^4}\right]u = 0.$$

From equation (58), $A = 3/16$, and we arbitrarily choose $2B = B_1 a_1^4$, and $C = C_1 a_1^4$.

Substitution of these values into equation (124) and slight rearrangement gives us the following expression:

$$(125) \qquad \frac{d^2u}{dz^2} + \operatorname*{Lim}_{a_1 \to \infty} \left[\frac{3z^2/16a_1^4 + B_1 z + C_1}{(z/a_1 - 1)^4}\right]u = 0.$$

Letting $a_1 \to \infty$ in equation (125) yields:

$$(126) \qquad \frac{d^2u}{dz^2} + (B_1 z + C_1)u = 0.$$

We now make the following dependent variable and independent variable transformations:

$$u = (B_1 z + C_1)^{1/2} v, \text{ and} \tag{127}$$

$$B_1 z + C_1 = (3B_1 w/2)^{2/3}. \tag{128}$$

Now from equation (127) we get:

$$\frac{du}{dz} = (B_1 z + C_1)^{1/2} \frac{dv}{dz} + \frac{B_1 v}{2} (B_1 z + C_1)^{-1/2}, \text{ and} \tag{129}$$

$$\frac{d^2u}{dz^2} = (B_1 z + C_1)^{1/2} \frac{d^2v}{dz^2} + \frac{B_1}{2} (B_1 z + C_1)^{-1/2} \frac{dv}{dz}$$

$$- \frac{B_1^2 v}{4} (B_1 z + C_1)^{-3/2} + \frac{B_1}{2} (B_1 z + C_1)^{-1/2} \frac{dv}{dz}, \text{ or}$$

$$\frac{d^2u}{dz^2} = (B_1 z + C_1)^{1/2} \frac{d^2v}{dz^2} + B_1 (B_1 z + C_1)^{-1/2} \frac{dv}{dz} \tag{130}$$

$$- \frac{B_1^2 v}{4} (B_1 z + C_1)^{-3/2}.$$

We now substitute equations (130) and (127) into equation (126) to get:

$$(B_1 z + C_1)^{1/2} \frac{d^2v}{dz^2} + B_1 (B_1 z + C_1)^{-1/2} \frac{dv}{dz} \tag{131}$$

$$- \frac{B_1^2 v}{4} (B_1 z + C_1)^{-3/2} + (B_1 z + C_1)^{3/2} v = 0.$$

Now from equation (128) we have:

$$B_1 \frac{dz}{dw} = \frac{2}{3}(3B_1w/2)^{-1/3}(3B_1/2) = (3B_1w/2)^{-1/3}, \text{ or}$$

(132) $\frac{dw}{dz} = (3B_1w/2)^{1/3}$. Hence it follows that

(133) $\frac{dv}{dz} = (3B_1w/2)^{1/3}\frac{dv}{dw}$.

From equation (133) we get:

(134) $\frac{d^2v}{dz^2} = (3B_1w/2)^{1/3}\frac{d^2v}{dw^2}\cdot\frac{dw}{dz} + \frac{B_1}{2}(3B_1w/2)^{-2/3}(3B_1w/2)^{1/3}\frac{dv}{dw}$.

Substitution of equations (128), (133), and (134) into equation (131) gives us the following:

$$(\tfrac{3}{2}B_1w)\frac{d^2v}{dw^2} + \frac{B_1}{2}\frac{dv}{dw} + B_1\frac{dv}{dw} - \frac{B_1^2v}{4}(3B_1w/2)^{-1}$$

$$+ (3B_1w/2)\,v = 0, \text{ or}$$

(135) $(\tfrac{3}{2}B_1w)\frac{d^2v}{dw^2} + \frac{3}{2}B_1\frac{dv}{dw} + [3B_1w/2 - B_1/6w]v = 0.$

Multiplying both members of equation (135) by $2w/3B_1$ gives us the final result:

(136) $w^2\frac{d^2v}{dw^2} + w\frac{dv}{dw} + (w^2 - 1/9)v = 0.$

Equation (136) is Stokes' equation and has been classified by Ince [5, p. 497 ] as an equation of the [0,0,1] class.

The foregoing analysis has been an introduction to second order linear ordinary differential equations with five regular singular points. The derivation and classification of certain confluent forms has also been given.

## CHAPTER III

## SOME CONSIDERATIONS OF A DIFFERENTIAL EQUATION WITH n-REGULAR SINGULAR POINTS

Consider the differential equation

$$(1)\qquad \frac{d^2u}{dz^2} + \sum_{r=1}^{n} \frac{1-\alpha_r-\beta_r}{z-a_r}\,\frac{du}{dz} + \sum_{r=1}^{n} \frac{\alpha_r\beta_r}{(z-a_r)^2} + \sum_{r=1}^{n} \frac{J_r}{z-a_r}\,u = 0.$$

All points including infinity are ordinary points, except the points $a_1, a_2, \ldots, a_n$. The points $a_1, a_2, \ldots, a_n$ are regular singular points with exponents $\alpha_1, \beta_1; \alpha_2, \beta_2; \ldots; \alpha_n, \beta_n$, respectively.

In order to assist in the future development of this study, we wish to show that

$$(2)\qquad \sum_{r=1}^{n} (\alpha_r + \beta_r) = n - 2,$$

$$(3)\qquad \sum_{r=1}^{n} J_r = 0,$$

$$(4)\qquad \sum_{r=1}^{n} (a_r J_r + \alpha_r\beta_r) = 0, \text{ and}$$

$$(5)\qquad \sum_{r=1}^{n} (a_r^2 J_r + 2a_r\alpha_r\beta_r) = 0.$$

Let us write equation (1) as follows:

$$(6)\quad \frac{d^2u}{dz^2} + \left[\frac{E_o+E_1z + \ldots + E_{n-1}z^{n-1}}{(z-a_1)(z-a_2)\cdots(z-a_n)}\right]\frac{du}{dz}$$

$$+ \left[\frac{F_o+F_1z + \ldots + F_{2n-2}z^{2n-2}}{(z-a_1)^2(z-a_2)^2\cdots(z-a_n)^2}\right] u = 0, \text{ or alternately,}$$

$$(7)\quad \frac{d^2u}{dz^2} + p(z)\frac{du}{dz} + q(z)\,u = 0, \text{ where}$$

$$(8)\quad p(z) = \frac{E_o+E_1z + \ldots + E_{n-1}z^{n-1}}{(z-a_1)(z-a_2)\cdots(z-a_n)},$$

$$(9)\quad q(z) = \frac{F_o+F_1z + \ldots + F_{2n-2}z^{2n-2}}{(z-a_1)^2(z-a_2)^2\cdots(z-a_n)^2}, \text{ and}$$

$E_i$, $(i = 0,1,2,\ldots,(n-1))$, and $F_i$, $(i = 0,1,2,\ldots,(2n-2))$ are constants to be evaluated.

Expanding equations (8) and (9) in partial fractions yields:

$$(10)\quad p(z) = \frac{G_1}{z-a_1} + \frac{G_2}{z-a_2} + \ldots + \frac{G_n}{z-a_n}, \text{ and}$$

$$(11)\quad q(z) = \frac{H_1}{(z-a_1)^2} + \frac{H_2}{(z-a_2)^2} + \ldots + \frac{H_n}{(z-a_n)^2}$$

$$+ \frac{J_1}{z-a_1} + \frac{J_2}{z-a_2} + \ldots + \frac{J_n}{z-a_n},$$

where the constants $G_r$, $H_r$ and $J_r$, $(r = 1,2,\ldots,n)$ will be evaluated presently.

Substituting equations (10) and (11) into equation (6) we get:

$$(12)\qquad \frac{d^2u}{dz^2} + \left[\frac{G_1}{z-a_1} + \frac{G_2}{z-a_2} + \cdots + \frac{G_n}{z-a_n}\right]\frac{du}{dz}$$

$$+ \left[\frac{H_1}{(z-a_1)^2} + \frac{H_2}{(z-a_2)^2} + \cdots + \frac{H_n}{(z-a_n)^2}\right.$$

$$\left. + \frac{J_1}{z-a_1} + \frac{J_2}{z-a_2} + \cdots + \frac{J_n}{z-a_n}\right]u = 0.$$

We now assume a solution of equation (11), in a neighborhood of $z = a_1$, of the form:

$$(13)\qquad u = \sum_{m=0}^{\infty} c_m(z-a_1)^{\lambda+m};\quad c_0 \neq 0.$$

Differentiating equation (13) with respect to z yields:

$$(14)\qquad u' = c_0\lambda(z-a_1)^{\lambda-1} + \sum_{m=1}^{\infty} c_m(\lambda+m)(z-a_1)^{\lambda+m-1}, \text{ and}$$

differentiating equation (14) with respect to z yields:

$$(15)\qquad u'' = c_0\lambda(\lambda-1)(z-a_1)^{\lambda-2} + \sum_{m=1}^{\infty} c_m(\lambda+m)(\lambda+m-1)(z-a_1)^{\lambda+m-2}.$$

We now substitute equations (13), (14), and (15) into equation (12) to get:

$$(16)\quad c_o\lambda(\lambda-1)(z-a_1)^{\lambda-2} + \sum_{m=1}^{\infty} c_m(\lambda+m)(\lambda+m-1)(z-a_1)^{\lambda+m-2}$$

$$+\left[\sum_{r=1}^{n} \frac{G_r}{z-a_r}\right] \cdot \left[c_o\lambda(z-a_1)^{\lambda-1} + \sum_{m=1}^{\infty} c_m(\lambda+m)(z-a_1)^{\lambda+m-1}\right]$$

$$+\left[\sum_{r=1}^{n} \frac{H_r}{(z-a_r)^2} + \sum_{r=1}^{n} \frac{J_r}{z-a_r}\right]$$

$$\cdot\left[c_o(z-a_1)^{\lambda} + \sum_{m=1}^{\infty} c_m(z-a_1)^{\lambda+m}\right] = 0.$$

We recall that $a_r, (r = 1,2,\ldots,n)$ are finite distinct regular singular points. Also we are expanding in ascending powers of $(z-a_1)$. Therefore, we obtain the indicial equation by equating to zero the coefficient of $(z-a_1)^{\lambda-2}$. Then

$$(17)\quad c_o\lambda(\lambda-1) + c_o\lambda g(G_1,G_2,\ldots,G_n) + c_o h(H_1,H_2,\ldots,H_n) = 0,$$

where $g(G_1,G_2,\ldots,G_n)$ and $h(H_1,H_2,\ldots,H_n)$ are constants which we now determine.

From equation (16), let

$$(18)\quad S_1 = \left[\sum_{r=1}^{n} \frac{G_r}{z-a_r}\right] \cdot [\, c_o\lambda(z-a_1)^{\lambda-1}], \text{ or}$$

$$(19)\quad S_1 = [\, \frac{G_1}{z-a_1}\,] \cdot [\, c_o\lambda(z-a_1)^{\lambda-1}\,]$$

$$+\left[\sum_{r=2}^{n} \frac{G_r}{z-a_r}\right] \cdot [\, c_o\lambda(z-a_1)^{\lambda-1}\,].$$

Now write

(20) $$S_2 = \left[\sum_{r=2}^{n} \frac{G_r}{z-a_r}\right] \cdot \left[c_o\lambda(z-a_1)^{\lambda-1}\right]$$ as follows:

$$S_2 = \left[\sum_{r=2}^{n} \frac{G_r}{z-a_1 + a_1-a_r}\right] \cdot \left[c_o\lambda(z-a_1)^{\lambda-1}\right]$$

$$= \left[c_o\lambda(z-a_1)^{\lambda-1}\right]\left[\sum_{r=2}^{n} \frac{G_r}{a_1-a_r} \cdot \frac{1}{\frac{z-a_1}{a_1-a_r} + 1}\right].$$

(21) $$S_2 = \left[c_o\lambda(z-a_1)^{\lambda-1}\right] \cdot \sum_{r=2}^{n}\left[\frac{G_r}{a_1-a_r} \sum_{j=0}^{\infty}(-1)^j\left(\frac{z-a_1}{a_1-a_r}\right)^j\right].$$

Substituting equation (21) into equation (19) yields:

(22) $$S_1 = \left[c_o\lambda(z-a_1)^{\lambda-1}\right]\left\{\frac{G_1}{z-a_1} + \sum_{r=2}\left[\frac{G_r}{a_1-a_r} \sum_{j=0}^{\infty}(-1)^j\left(\frac{z-a_1}{a_1-a_r}\right)^j\right]\right\},$$

from which we observe that the least exponent on $(z-a_1)$ is $(\lambda-2)$, and the corresponding coefficient of $(z-a_1)^{\lambda-2}$ is $c_o\lambda G_1$. We thus conclude that $\sum_{r=2}^{n} \frac{G_r}{z-a_r}$ contributes nothing to the indicial equation. The sole contribution comes from $\frac{G_1}{z-a_1}$, hence in equation (17), $g(G_1,G_2,\ldots,G_n) = G_1$, and equation (17) now becomes:

(23) $c_o\lambda(\lambda-1) + c_o\lambda G_1 + c_o h(H_1, H_2, \ldots H_n) = 0$, or

since $c_o \neq 0$ by assumption, equation (23) becomes:

(24) $\lambda(\lambda-1) + \lambda G_1 + h(H_1, H_2, \ldots H_n) = 0$

We note in passing that from equation (16)

(25) $S_3 = \left[\sum_{r=1}^{n} \frac{G_r}{z-a_r}\right] \cdot \left[\sum_{m=1}^{\infty} c_m(\lambda+m)(z-a_1)^{\lambda+m-1}\right]$ contributes nothing to the indicial equation since the least exponent on $(z-a_1)$ in the above expression is $\lambda-1 > \lambda-2$.

Returning to equation (16), we let

(26) $S_4 = c_o(z-a_1)^{\lambda} \sum_{r=1}^{n} \frac{H_r}{(z-a_r)^2}$,

(27) $S_5 = \left[\sum_{r=1}^{n} \frac{H_r}{(z-a_r)^2}\right] \cdot \left[\sum_{m=1}^{\infty} c_m(z-a_1)^{\lambda+m}\right]$,

(28) $S_6 = c_o(z-a_1)^{\lambda} \sum_{r=1}^{n} \frac{J_r}{z-a_r}$, and

(29) $S_7 = \left[\sum_{r=1}^{n} \frac{J_r}{z-a_r}\right] \cdot \left[\sum_{m=1}^{\infty} c_m(z-a_1)^{\lambda+m}\right]$.

We observe that the least exponent on $(z-a_1)$ in equation (27) is $\lambda-1 > \lambda-2$. Similarly the least exponent on $(z-a_1)$ in equation (28) is $\lambda-1 > \lambda-2$, and in equation (29) the least exponent on $(z-a_1)$ is $\lambda > \lambda-2$, hence equations

(27) - (29) contribute nothing to the indicial equation. The only contribution to the indicial equation comes from equation (26).

Let us write equation (26) as follows:

$$S_4 = c_o(z-a_1)^{\lambda} \frac{H_1}{(z-a_1)^2} + c_o(z-a_1)^{\lambda} \sum_{r=2}^{n} \frac{H_r}{(z-a_r)^2}, \text{ or}$$

$$S_4 = c_o(z-a_1)^{\lambda-2} \cdot H_1 + c_o(z-a_1)^{\lambda} \sum_{r=2}^{n} \frac{H_r}{(z-a_1 + a_1-a_r)^2}.$$

This may be written as follows:

$$S_4 = c_o H_1(z-a_1)^{\lambda-2} + c_o(z-a_1)^{\lambda} \sum_{r=2}^{n} \left[ \frac{H_r}{(a_1-a_r)^2} \cdot \frac{1}{\left(1+\frac{z-a_1}{a_1-a_r}\right)^2} \right].$$

We may write the above expression for $S_4$ as follows:

$$(30) \quad S_4 = c_o H_1(z-a_1)^{\lambda-2} + c_o(z-a_1)^{\lambda} \sum_{r=2}^{n} \frac{H_r}{(a_1-a_r)^2} \left[ \sum_{j=0}^{\infty} (-1)^j \left(\frac{z-a_1}{a_1-a_r}\right)^j \right]^2 .$$

Now the least exponent on $(z-a_1)$ in the expression

$$c_o(z-a_1)^{\lambda} \sum_{r=2}^{n} \frac{H_r}{(a_1-a_r)^2} \left[ \sum_{j=0}^{\infty} (-1)^j \left(\frac{z-a_1}{a_1-a_r}\right)^j \right]^2 \text{ is } \lambda,$$

which can be found by actual expansion, so that this expression contributes nothing to the indicial equation. Consequently, the only contribution to the indicial equation,

insofar as equations (26) - (30) are concerned, comes only from the first part of equation (30) and is the coefficient of $(z-a_1)^{\lambda-2}$, that is, $(c_o)(H_1)$.

Returning now to equation (23) we note that

$c_o h(H_1, H_2, \ldots, H_n) = c_o H_1$, or

$h(H_1, H_2, \ldots, H_n) = H_1$, thus equation (24) becomes:

(31) $\lambda(\lambda-1) + \lambda G_1 + H_1 = 0$, or in alternate form:

(32) $\lambda^2 + \lambda(G_1-1) + H_1 = 0$.

Equation (28) was derived upon expanding in a neighborhood of $z = a_1$. Analogous reasoning shows that expansion in a neighborhood of any of the regular singular points $z = a_r$, would give an indicial equation of the form:

(33) $\lambda^2 + \lambda(G_r-1) + H_r = 0; (r = 1,2,\ldots,n)$.

Let the values of $\lambda$ which satisfy equation (33) be $\lambda_1$ and $\lambda_2$. We then have

(34) $\lambda_1 + \lambda_2 = 1-G_r$, and

(35) $\lambda_1\lambda_2 = H_r$. But we are given that the exponents at $a_r$ are $\alpha_r$ and $\beta_r$, respectively, hence equations (34) and (35) become:

$\lambda_1 + \lambda_2 = 1-G_r = \alpha_r + \beta_r$, or

(36) $G_r = 1-\alpha_r-\beta_r$. Similarly,

(37) $H_r = \alpha_r\beta_r$.

Substituting equations (36) and (37) into equation (12) yields the following differential equation:

$$(38)\qquad \frac{d^2u}{dz^2} + \left[\sum_{r=1}^{n} \frac{1-\alpha_r-\beta_r}{z-a_r}\right]\frac{du}{dz} + \left[\sum_{r=1}^{n} \frac{\alpha_r\beta_r}{(z-a_r)^2} + \sum_{r=1}^{n} \frac{J_r}{z-a_r}\right]u = 0.$$

We now make use of the requirement that infinity be an ordinary point of equation (38).

We make the independent variable transformation:

(39) $z = \frac{1}{w}$, substitute into equation (38), and require $w = 0$ to be an ordinary point of the resulting equation.

The equation resulting from the above variable transformation was derived in Chapter I, and was shown to be:

$$(40)\qquad \frac{d^2u}{dw^2} + \left[\frac{2}{w} - \frac{1}{w^2}\,p\left(\frac{1}{w}\right)\right]\frac{du}{dw} + \frac{1}{w^4}\,q\left(\frac{1}{w}\right) = 0.$$

It was further shown that if $w = 0$ is to be an ordinary point, then both

$$\left[\frac{2}{w} - \frac{1}{w^2}\,p\left(\frac{1}{w}\right)\right] \quad \text{and} \quad \frac{1}{w^4}\,q\left(\frac{1}{w}\right) \quad \text{must be analytic at } w = 0.$$

We now expand $p(\frac{1}{w})$ in an ascending power series about $w = 0$ of the form:

$$(41)\qquad p\left(\frac{1}{w}\right) = 2w + A_2w^2 + \cdots + A_nw^n + \cdots, \text{ and hence}$$

$$(42)\qquad \left[\frac{2}{w} - \frac{1}{w^2}\,p\left(\frac{1}{w}\right)\right] = \frac{2}{w} - \left[\frac{2}{w} + A_2 + \cdots + A_nw^{n-2} + \cdots\right]$$

$$= -A_2 - A_3w - \cdots - A_nw^{n-2} - \cdots, \text{ which}$$

is analytic at $w = 0$, provided $A_2$ is finite.

This implies $p(\frac{1}{w})$ is finite at $w = 0$, or

(43) $$p(z) = \frac{2}{z} + \frac{A_2}{z^2} + \cdots + \frac{A_n}{z^n} + \cdots \text{ is finite at infinity.}$$

Now from equation (38) we notice that

(44) $$p(z) = \sum_{r=1}^{n} \frac{(1-\alpha_r-\beta_r)}{z-a_r}$$

$$= \frac{1}{z} \sum_{r=1}^{n} \frac{(1-\alpha_r-\beta_r)}{1 - \frac{a_r}{z}}$$

$$= \frac{1}{z} \sum_{r=1}^{n} (1-\alpha_r-\beta_r) \cdot \left[1 + \left(\frac{a_r}{z}\right) + \left(\frac{a_r}{z}\right)^2 + \cdots + \left(\frac{a_r}{z}\right)^j + \cdots\right], \text{ or}$$

(45) $$p(z) = \frac{1}{z} \sum_{r=1}^{n} (1-\alpha_r-\beta_r) + \frac{1}{z} \sum_{r=1}^{n} \left[(1-\alpha_r-\beta_r) \sum_{j=1}^{\infty} \left(\frac{a_r}{z}\right)^j\right].$$

Now upon comparing equations (43) and (45) and equating the coefficients of like powers of $\frac{1}{z}$ we get:

$$\sum_{r=1}^{n} (1-\alpha_r-\beta_r) = 2, \text{ or finally,}$$

(46) $$\sum_{r=1}^{n} (\alpha_r + \beta_r) = n-2.$$

Similar to the preceding analysis, we expand $q(\frac{1}{w})$ about $w = 0$ in an ascending power series thus:

(47) $q(\frac{1}{w}) = B_1 w + B_2 w^2 + \cdots + B_n w^n + \cdots$ , which is

finite at w = 0. From equation (47) we have

(48) $\frac{1}{w^4} q(\frac{1}{w}) = \frac{B_1}{w^3} + \frac{B_2}{w^2} + \frac{B_3}{w} + B_4 + B_5 w + \cdots + B_n w^{n-4} + \cdots$ .

In order for $\frac{1}{w^4} q(\frac{1}{w})$ to be analytic at w = 0, $B_1 = B_2 = B_3 = 0$, in which case equation (47) becomes:

(49) $q(\frac{1}{w}) = B_4 w^4 + B_5 w^5 + \cdots + B_n w^n + \cdots$ , and

since $z = \frac{1}{w}$, we have:

(50) $q(z) = \frac{B_4}{z^4} + \frac{B_5}{z^5} + \cdots + \frac{B_n}{z^n} + \cdots$ .

Since $q(\frac{1}{w})$, as given by equation (49), is finite at w = 0, then q(z), as given by equation (50), is finite at infinity. This completes the restrictions under which infinity will be an ordinary point of equation (38).

From equation (38) we notice that

(51) $q(z) = \sum_{r=1}^{n} \frac{\alpha_r \beta_r}{(z-a_r)^2} + \sum_{r=1}^{n} \frac{J_r}{(z-a_r)}$; ( r = 1,2,$\cdots$,n).

Equation (51) may be written thus:

$$q(z) = \frac{1}{z^2} \sum_{r=1}^{n} \frac{\alpha_r \beta_r}{\left(1 - \frac{a_r}{z}\right)^2} + \frac{1}{z} \sum_{r=1}^{n} \frac{J_r}{1 - \frac{a_r}{z}}$$

$$= \frac{1}{z^2} \sum_{r=1}^{n} \alpha_r \beta_r \left[1 + \left(\frac{a_r}{z}\right) + \left(\frac{a_r}{z}\right)^2 + \cdots + \left(\frac{a_r}{z}\right)^j + \cdots\right]^2$$

$$+ \frac{1}{z} \sum_{r=1}^{n} J_r \left[1 + \left(\frac{a_r}{z}\right) + \left(\frac{a_r}{z}\right)^2 + \cdots + \left(\frac{a_r}{z}\right)^j + \cdots\right].$$

$$q(z) = \frac{1}{z^2} \sum_{r=1}^{n} \alpha_r \beta_r \left[1 + \left(\frac{a_r}{z}\right)^2 + \left(\frac{2a_r}{z}\right) + \left(\frac{2a_r^2}{z^2}\right) + \cdots\right]$$

$$+ \frac{1}{z} \sum_{r=1}^{n} J_r \left[1 + \left(\frac{a_r}{z}\right) + \left(\frac{a_r}{z}\right)^2 + \cdots\right].$$

(52) $$q(z) = \sum_{r=1}^{n} \frac{\alpha_r \beta_r}{z^2} + 2 \sum_{r=1}^{n} \frac{a_r \alpha_r \beta_r}{z^3} + \sum_{r=1}^{n} \frac{J_r}{z} + \sum_{r=1}^{n} \frac{a_r J_r}{z^2}$$

$$+ \sum_{r=1}^{n} \frac{a_r^2 J_r}{z^3} + O\left(\frac{1}{z^4}\right).$$

Finally we may write equation (52) as follows:

(53) $$q(z) = \frac{1}{z} \sum_{r=1}^{n} J_r + \frac{1}{z^2} \sum_{r=1}^{n} (\alpha_r \beta_r + a_r J_r)$$

$$+ \frac{1}{z^3} \sum_{r=1}^{n} (2a_r \alpha_r \beta_r + a_r^2 J_r) + O\left(\frac{1}{z^4}\right).$$

We now compare equation (53) with equation (50) and equate coefficients of like powers of $\frac{1}{z}$ to obtain the following set of equations:

$$(54) \qquad \sum_{r=1}^{n} J_r = 0,$$

$$(55) \qquad \sum_{r=1}^{n} (\alpha_r\beta_r + a_rJ_r) = 0, \text{ and}$$

$$(56) \qquad \sum_{r=1}^{n} (2a_r\alpha_r\beta_r + a_r^2J_r) = 0.$$

Equations (46), and (54) - (56) are the relationships we desired. They will be used in the derivation of Heun's differential equation [14, p. 576] in Chapter IV, and in the derivation of a general linear ordinary differential equation of second order with five regular singular points as shown in Chapter V.

## CHAPTER IV

## A DISCUSSION OF HEUN'S DIFFERENTIAL EQUATION

We will now apply the relationships derived in Chapter III to the derivation and solution, in a general case, of Heun's differential equation [14, p. 576]. An intensive study of Heun's differential equation was recently completed by Neff [9].

Consider the differential equation:

$$(1)\qquad \frac{d^2u}{dz^2} + \left[\sum_{r=1}^{n} \frac{1-\alpha_r-\beta_r}{z-a_r}\right]\frac{du}{dz} + \left[\sum_{r=1}^{n} \frac{\alpha_r\beta_r}{(z-a_r)^2} + \sum_{r=1}^{n} \frac{J_r}{z-a_r}\right]u = 0.$$

In Chapter III, we derived the following relationships:

$$(2)\qquad \sum_{r=1}^{n} (\alpha_r + \beta_r) = n-2;$$ where n is the number of regular singular points.

$$(3)\qquad \sum_{r=1}^{n} J_r = 0,$$

$$(4)\qquad \sum_{r=1}^{n} (a_r J_r + \alpha_r\beta_r) = 0, \text{ and}$$

$$(5)\qquad \sum_{r=1}^{n} (a_r^2 J_r + 2a_r\alpha_r\beta_r) = 0.$$

We arbitrarily let

$a_1 = 0$; $a_2 = 1$; $a_3 = a$; $a_4 = b$;

$\alpha_1 = 0$; $\alpha_2 = 0$; $\alpha_3 = 0$; $\alpha_4 = \alpha$;

$\beta_1 = 1-\gamma$; $\beta_2 = 1-\delta$; $\beta_3 = \gamma+\delta-\alpha-\beta$; $\beta_4 = \beta$.

Substitute the above set of equations into equation (1) to get:

$$(6)\quad \frac{d^2u}{dz^2} + \left[\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{1+\alpha+\beta-\gamma-\delta}{z-a} + \frac{1-\alpha-\beta}{z-b}\right]\frac{du}{dz}$$

$$+\left[\frac{\alpha\beta}{(z-b)^2} + \frac{J_1}{z} + \frac{J_2}{z-1} + \frac{J_3}{z-a} + \frac{J_4}{z-b}\right] u = 0.$$

We also note that the above set of equations when compared with equations (2) - (5) yield the following expressions:

$$(7)\quad \alpha + (1-\gamma) + (1-\delta) + (\gamma+\delta-\alpha-\beta) + \beta = 2,$$

$$(8)\quad J_1 + J_2 + J_3 + J_4 = 0,$$

$$(9)\quad J_2 + aJ_3 + bJ_4 + \alpha\beta = 0, \text{ and}$$

$$(10)\quad J_2 + a^2J_3 + b^2J_4 + 2b\alpha\beta = 0.$$

To continue with the derivation of Heun's differential equation, we multiply both members of equation (6) by $(z)(z-1)(z-a)$ to get:

$$(11)\quad z(z-1)(z-a)\frac{d^2u}{dz^2} + [\gamma(z-1)(z-a) + z(z-a)\delta + z(z-1)(\alpha+\beta-\gamma-\delta+1) + z(z-1)(z-a)(1-\alpha-\beta)/(z-b)]\frac{du}{dz} + [z(z-1)(z-a)(\alpha\beta)/(z-b)^2 + (z-1)(z-a)J_1 + z(z-a)J_2 + z(z-1)J_3 + z(z-1)(z-a)(J_4)/(z-b)]\,u = 0.$$

Let $C_2$ be the coefficient of $\frac{du}{dz}$, that is,

$$C_2 = (z^2-z-az+a)\gamma + \delta(z^2-az) + (\alpha+\beta-\gamma-\delta+1)(z^2-z) + \frac{z(z-1)(z-a)(1-\alpha-\beta)}{z-b}$$

$$= z^2(\gamma+\delta+\alpha+\beta-\gamma-\delta+1) + z(-\gamma-a\gamma-a\delta-\alpha-\beta+\gamma+\delta+1) + \frac{z(z-1)(z-a)(1-\alpha-\beta)}{z-b}\text{, or finally we have:}$$

$$(12)\quad C_2 = z^2(1+\alpha+\beta) - z(1+\alpha+\beta-\delta+a\delta+a\gamma) + a\gamma + \frac{z(z-1)(z-a)(1-\alpha-\beta)}{z-b}.$$

In equation (11), we let $C_3$ be the coefficient of u, that is,

$$C_3 = \frac{1}{(z-b)^2}[z(z-1)(z-a)(\alpha\beta) + (z-b)^2(z-1)(z-a)(J_1) + z(z-a)(z-b)^2(J_2) + z(z-1)(z-b)^2(J_3) + z(z-1)(z-a)(z-b)(J_4)].$$

$$C_3 = \frac{1}{(z-b)^2}\{(z^3-az^2-z^2+az)(\alpha\beta)$$

$$+ [z^4-z^3(a+2b+1) + z^2(a+2ab+2b+b^2)$$

$$- z(2ab+ab^2+b^2) + ab^2]J_1$$

$$+ [z^4-z^3(2b+a) + z^2(b^2+2ab) - z(ab^2)]J_2$$

$$+ [z^4-z^3(2b+1) + z^2(b^2+2b) - zb^2]J_3$$

$$+ [z^4-z^3(a+b+1) + z^2(ab+a+b) - z(ab)]J_4\}.$$

We now collect the coefficients of like powers of z to get:

$$(13) \quad C_3 = \frac{1}{(z-b)^2}\{z^4[J_1+J_2+J_3+J_4] + z^3[\alpha\beta -(a+2b+1)J_1$$

$$- (a+2b)J_2 - (2b+1)J_3 - (a+b+1)J_4]$$

$$+ z^2[-a\alpha\beta-\alpha\beta+(a+2ab+2b+b^2)J_1$$

$$+ (2ab+b^2)J_2 + (b^2+2b)J_3 + (a+b+ab)J_4]$$

$$- z[-a\alpha\beta+(2ab+ab^2+b^2)J_1+ab^2J_2+b^2J_3+abJ_4]$$

$$+ ab^2J_1\}.$$

Applying the restriction given by equation (8), we see that the coefficient of $z^4$ vanishes. We now write the coefficient of $z^3$ as follows:

$z^3$: $\alpha\beta - (a+2b+1)J_1 - (a+2b+1-1)J_2 - (a+2b+1-a)J_3$

$- (a+2b+1-b)J_4$

$= - (a+2b+1)(J_1+J_2+J_3+J_4) + (\alpha\beta+J_2+aJ_3+bJ_4)$

$= 0$ by virtue of equations (8) and (9).

We now simplify the coefficient of $z^2$ thus:

$z^2$: $(a+2ab+2b+b^2)J_1 + (a+2ab+2b+b^2-a-2b)J_2$

$+ (a+2ab+2b+b^2-a-2ab)J_3 + (a+2ab+2b+b^2-b-ab-b^2)J_4$

$- a\alpha\beta - \alpha\beta$

$= (a+2ab+2b+b^2)(J_1+J_2+J_3+J_4) - (a+2b)J_2$

$- (a+2ab)J_3 - (b+ab+b^2)J_4 - a\alpha\beta - \alpha\beta$

$= (a+2ab+2b+b^2)(J_1+J_2+J_3+J_4) - (a+2b+1-1)J_2$

$- a(a+2b+1-a)J_3 - b(a+2b+1-b)J_4 - (a+2b+1-2b)\alpha\beta$

$= (a+2ab+2b+b^2)(J_1+J_2+J_3+J_4) - (a+2b+1)$

$\cdot(\alpha\beta+J_2+aJ_3+bJ_4) + (J_2+a^2J_3+b^2J_4+2b\alpha\beta)$

$= 0$ by virtue of equations (8), (9) and (10).

The expression for $C_3$ has now been reduced to the following:

$$C_3 = \frac{-z}{(z-b)^2}\left[a\alpha\beta + (2ab+ab^2+b^2)J_1 + ab^2J_2+b^2J_3+abJ_4\right] + \frac{ab^2J_1}{(z-b)^2}.$$

We now proceed to simplify the coefficient of $-z$ thus:

$$-z: \quad (ab^2+b^2+ab+ab)J_1 + (ab^2+b^2+ab-b^2-ab)J_2$$
$$+ (ab^2+b^2+ab-ab^2-ab)J_3 + (ab^2+b^2+ab-b^2-ab^2)J_4$$
$$= (ab^2+b^2+ab)(J_1+J_2+J_3+J_4) + abJ_1 - (ab+b^2)J_2$$
$$- (ab+ab^2)J_3 - (ab^2+b^2)J_4 + a\alpha\beta, \text{ which upon}$$

applying equation (8) and rearranging somewhat becomes:

$$- (b^2+ab+b-b)J_2 - a(b^2+ab+b-ab)J_3$$
$$- b(b+ab+b^2-b^2)J_4 + abJ_1 + a\alpha\beta$$
$$= - (b^2+ab+b-b^2-ab-b-a)\alpha\beta - a(b^2+ab+b-ab)J_3$$
$$- b(b+ab+b^2-b^2)J_4 + abJ_1 - (b^2+ab+b-b)J_2$$
$$= - (b^2+ab+b)(\alpha\beta+J_2+aJ_3+bJ_4) + abJ_1$$
$$+ b(J_2+a^2J_3+b^2J_4) + (b^2+ab+a+b)\alpha\beta.$$

Utilizing equation (9), the above expression simplifies, upon slight rearrangement to:

$$-z: \quad b(J_2+a^2J_3+b^2J_4+2b\alpha\beta) + abJ_1 + (a+b+ab-b^2)\alpha\beta$$
$$= abJ_1 + (a+b+ab-b^2)\alpha\beta, \text{ by virtue of equation (10).}$$

Equation (13) has now been reduced to the following:

$$(14) \quad C_3 = \frac{-z[abJ_1 + (a+b+ab-b^2)\alpha\beta] + ab^2J_1}{(z-b)^2} .$$

We now substitute equations (12) and (14) into equation (11) and express the differential equation as follows:

$$(15)\quad z(z-1)(z-a)\frac{d^2u}{dz^2} + [z^2(1+\alpha+\beta) - z(1+\alpha+\beta-\delta+a\delta+a\gamma) + a\gamma + z(z-1)(z-a)(1-\alpha-\beta)/(z-b)]\frac{du}{dz} - \frac{\{z[abJ_1+ (a+b+ab-b^2)\alpha\beta] - ab^2J_1\}}{(z-b)^2}u = 0.$$

Since infinity is a regular singular point, we let $b \to \infty$ to get:

$$(16)\quad z(z-1)(z-a)\frac{d^2u}{dz^2} + [z^2(1+\alpha+\beta) - z(1+\alpha+\beta-\delta+a\delta+a\gamma) + a\gamma]\frac{du}{dz} + (\alpha\beta z + aJ_1)\,u = 0.$$

In equation (16) we let

$\alpha\beta z + aJ_1 = \alpha\beta(z - q)$, whence

$aJ_1 = -\alpha\beta q$, or $q = -\frac{aJ_1}{\alpha\beta}$.

Substitution into equation (16) yields:

$$(17)\quad z(z-1)(z-a)\frac{d^2u}{dz^2} + [z^2(1+\alpha+\beta) - z(1+\alpha+\beta-\delta+a\delta+a\gamma)+a\gamma]\frac{du}{dz} + \alpha\beta(z - q)\,u = 0.$$

Equation (17) is Heun's differential equation and has four regular singular points at $z = 0,1,a$, and $\infty$.

We will now solve equation (17) in a general case, using the method of Frobenius [3].

Assume a general solution to equation (17) to be of the form:

(18) $u = c_o z^{\lambda} + c_1 z^{\lambda+1} + \cdots + c_{r-1} z^{\lambda+r-1} + c_r z^{\lambda+r}$

$+ c_{r+1} z^{\lambda+r+1} + \cdots$ , where $c_o \neq 0$. Then

(19) $\frac{du}{dz} = c_o \lambda z^{\lambda-1} + c_1(\lambda+1) z^{\lambda} + \cdots + c_{r-1}(\lambda+r-1) z^{\lambda+r-2}$

$+ c_r(\lambda+r) z^{\lambda+r-1} + c_{r+1}(\lambda+r+1) z^{\lambda+r} + \cdots$ , and

(20) $\frac{d^2u}{dz^2} = c_o \lambda(\lambda-1) z^{\lambda-2} + c_1(\lambda+1)(\lambda) z^{\lambda-1} + \cdots$

$+ c_{r-1}(\lambda+r-1)(\lambda+r-2) z^{\lambda+r-3} + c_r(\lambda+r)(\lambda+r-1) z^{\lambda+r-2}$

$+ c_{r+1}(\lambda+r+1)(\lambda+r) z^{\lambda+r-1} + \cdots$ .

We now substitute equations (18), (19) and (20) into equation (17), and deduce the indicial equation by setting the coefficient of $z^{\lambda-1}$ to zero, thus:

(21) $- a c_o \lambda(\lambda-1) + c_o a \lambda \gamma = 0$, where $a \neq 0$.

Solving equation (21) yields

$\lambda(\lambda-1) + \lambda\gamma = 0$, whence

(22) $\lambda = 0, 1 - \gamma$.

To obtain the recurrence equation connecting respective coefficients in equation (18), we set the coefficient of $z^{\lambda+r}$ to zero, that is:

$$(23)\quad c_{r-1}(\lambda+r-1)(\lambda+r-2) - c_r(a+1)(\lambda+r)(\lambda+r-1)$$

$$+ c_{r+1}(a)(\lambda+r+1)(\lambda+r) + c_{r-1}(1+\alpha+\beta)(\lambda+r-1)$$

$$- c_r(1+\alpha+\beta-\delta+a\gamma+a\delta)(\lambda+r) + c_{r+1}(a\gamma)(\lambda+r+1)$$

$$+ c_{r-1}(\alpha\beta) - c_r(\alpha\beta q) = 0 \text{ for } r = 0,1, \cdots .$$

We may rewrite equation (23) as follows:

$$(24)\quad c_{r+1}(a)(\lambda+r+1)(\lambda+r+\gamma) = c_r[(a+1)(\lambda+r)(\lambda+r-1) + \alpha\beta q$$

$$+ (1+\alpha+\beta - \delta+a\gamma+a\delta)(\lambda+r)]$$

$$- c_{r-1}[(\lambda+r-1)(\lambda+r-2) + (1+\alpha+\beta)(\lambda+r-1) + \alpha\beta].$$

Now from equation (22), we let $\lambda = 0$, so that equation (24) becomes:

$$(25)\quad c_{r+1}(a)(r+1)(r+\gamma) = c_r[(a+1)(r)(r-1) + \alpha\beta q$$

$$+ r(1+\alpha+\beta-\delta+a\gamma+a\delta)]$$

$$- c_{r-1}[(r-1)(r-2) + (1+\alpha+\beta)(r-1) + \alpha\beta].$$

We would like a general solution of equation (17) in a neighborhood of $z = 0$ to be of the following form when $\lambda = 0$:

$$(26)\quad u = c_o\left[1 + \alpha\beta \sum_{n=1}^{\infty} \frac{G_n(q)\cdot(z/a)^n}{n!(\gamma)(\gamma+1) \cdots (\gamma+n-1)}\right].$$

Setting r = 0 in equation (25), we get:

$c_1(a\gamma) = c_o(\alpha\beta q) - c_{-1}(1-\alpha)(1-\beta)$. But $c_{-1} = 0$,

hence $c_1 = \frac{\alpha\beta q c_o}{a\gamma}$. Substituting this expression for $c_1$ into equation (18) yields the following equation:

(27) $u = c_o[1 + \frac{\alpha\beta q}{a\gamma} z + c_2 z^2 + \cdots + c_n z^n + \cdots]$.

Comparing the coefficients of like powers of z in equations (26) and (27) we note

(28) $G_1(q) = q$.

Next we set r = 1 in equation (25) to get:

(29) $c_2(a)(2)(1+\gamma) = c_1(\alpha\beta q + 1+\alpha+\beta-\delta+a\gamma+a\delta) - c_o(\alpha\beta)$, where $c_1 = \frac{\alpha\beta q c_o}{a\gamma}$.

Solving equation (29) for $c_2$ yields:

(30) $c_2 = \frac{\alpha\beta[q(\alpha\beta q + 1+\alpha+\beta-\delta+a\gamma+a\delta) - a\gamma]c_o}{a^2 \cdot 2!(\gamma)(\gamma+1)}$.

Comparing $c_2$ from equation (30) with $G_2(q)$ from equation (26) we note:

(31) $G_2(q) = q(\alpha\beta q + 1+\alpha+\beta-\delta+a\gamma+a\delta) - a\gamma$.

Letting r = n in equation (25) and comparing $c_n$ so found with $G_n(q)$ from equation (26) reveals the following:

$$(32)\quad c_n = \frac{\alpha\beta G_n(q)\cdot c_o}{a^n\cdot n!(\gamma)(\gamma+1)\ \cdots\ (\gamma+n-1)}\ .$$

Substituting equation (32) into equation (25), with appropriate change of subscripts gives the following equation:

$$(33)\quad \frac{a(n+1)(n+\gamma)G_{n+1}(q)}{a^{n+1}\cdot(n+1)!(\gamma)(\gamma+1)\ \cdots\ (\gamma+n-1)(\gamma+n)}$$

$$= \frac{[(a+1)(n)(n-1) + \alpha\beta q + n(1+\alpha+\beta-\delta+a\gamma+a\delta)]G_n(q)}{a^n\cdot n!(\gamma)(\gamma+1)\ \cdots\ (\gamma+n-1)}$$

$$- \frac{[(n-1)(n-2) + (1+\alpha+\beta)(n-1) + \alpha\beta]G_{n-1}(q)}{a^{n-1}\cdot(n-1)!(\gamma)(\gamma+1)\ \cdots\ (\gamma+n-2)}\ .$$

Multiplying both members of equation (33) by $a^n\cdot n!(\gamma)(\gamma+1)\ \cdots\ (\gamma+n-1)$, and simplifying the result, gives us

$$(34)\quad G_{n+1}(q) = [(a+1)(n)(n-1) + \alpha\beta q + n(1+\alpha+\beta-\delta+a\gamma+a\delta)]G_n(q)$$

$$- [an(n-1+\alpha)(n-1+\beta)(n-1+\gamma)]G_{n-1}(q)\ .$$

Equation (34) may be used to compute coefficients $G_{n+1}(q)$ which may then be inserted into equation (26) to give a general solution of equation (17) in a neighborhood of $z = 0$, when $\lambda = 0$.

## CHAPTER V

## DERIVATION OF A GENERAL DIFFERENTIAL EQUATION WITH FIVE REGULAR SINGULAR POINTS

Consider the following differential equation:

$$(1)\qquad \frac{d^2u}{dz^2} + \sum_{r=1}^{n} \left(\frac{1-\alpha_r-\beta_r}{z-a_r}\right)\frac{du}{dz} + \left[\sum_{r=1}^{n} \frac{\alpha_r\beta_r}{(z-a_r)^2} + \sum_{r=1}^{n} \frac{D_r}{z-a_r}\right]u = 0.$$

Analysis in Chapter III revealed that for $J_r \equiv D_r$ ,

$$(2)\qquad \sum_{r=1}^{n} D_r = 0 ,$$

$$(3)\qquad \sum_{r=1}^{n} (a_r D_r + \alpha_r\beta_r) = 0 ,$$

$$(4)\qquad \sum_{r=1}^{n} (a_r^2 D_r + 2a_r\alpha_r\beta_r) = 0, \quad \text{and}$$

$$(5)\qquad \sum_{r=1}^{n} (\alpha_r + \beta_r) = n - 2.$$

Since we are interested in studying a second order linear ordinary differential equation with five distinct regular singular points, we choose n = 5. Then, following the scheme of Riemann [12], we denote a solution of equation (1) thus:

$$(6)\qquad u = P\left\{\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & z \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \end{matrix}\right\}.$$

As before, we note that the first row displays the regular singular points, rows two and three display respective exponents at these singular points, and column six contains the independent variable.

Since a linear transformation will take any three distinct points into any other three distinct points, there is no loss of generality if we let $a_1 = 0$, $a_2 = 1$, $a_3 = a$, $a_4 = b$, and $a_5 = c$. At the same time, we choose $\alpha_r = 0$ for $r = 1,2,3,4$, $\alpha_5 = \alpha$, $\beta_1 = 1 - \gamma$, $\beta_2 = 1 - \delta$, $\beta_3 = 1 - \epsilon$, $\beta_4 = 1 - \eta$, and $\beta_5 = \beta$. Substitution of this information into equation (6) yields:

$$(7) \qquad u = P \left\{ \begin{matrix} 0 & 1 & a & b & c & \\ 0 & 0 & 0 & 0 & \alpha & z \\ 1-\gamma & 1-\delta & 1-\epsilon & 1-\eta & \beta & \end{matrix} \right\}$$

where we shall presently let $c \to \infty$.

Equation (1) also becomes:

$$(8) \qquad \frac{d^2u}{dz^2} + \left[\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a} + \frac{\eta}{z-b} + \frac{1-\alpha-\beta}{z-c}\right]\frac{du}{dz}$$

$$+ \left[\frac{\alpha\beta}{(z-c)^2} + \frac{D_1}{z} + \frac{D_2}{z-1} + \frac{D_3}{z-a} + \frac{D_4}{z-b} + \frac{D_5}{z-c}\right]u = 0,$$

where from equations (2) - (5) we note that

$$(9) \qquad D_1 + D_2 + D_3 + D_4 + D_5 = 0,$$

$$(10) \qquad \alpha\beta + D_2 + aD_3 + bD_4 + cD_5 = 0,$$

$$(11) \qquad 2c\alpha\beta + D_2 + a^2D_3 + b^2D_4 + c^2D_5 = 0, \text{ and}$$

$$(12) \qquad \gamma + \delta + \epsilon + \eta - \alpha - \beta = 1.$$

Reducing equation (8) to a form which is more amenable to solution, we initially multiply both members by $z(z-1)(z-a)(z-b)$ to get:

$$(13)\quad z(z-1)(z-a)(z-b)\frac{d^2u}{dz^2} + \{\gamma(z-1)(z-a)(z-b)+\delta(z)(z-a)(z-b) + \epsilon(z)(z-1)(z-b)+\eta(z)(z-1)(z-a)+\frac{(1-\alpha-\beta)(z)(z-1)(z-a)(z-b)}{z-c}\}\frac{du}{dz} + \{\frac{\alpha\beta(z)(z-1)(z-a)(z-b)}{(z-c)^2} + D_1(z-1)(z-a)(z-b)+D_2(z)(z-a)(z-b) +D_3(z)(z-1)(z-b)+D_4(z)(z-1)(z-a)+\frac{D_5(z)(z-1)(z-a)(z-b)}{z-c}\}u = 0.$$

Let $K_2$ be the coefficient of $\frac{du}{dz}$, thus:

$$(14)\quad K_2 = \gamma(z-1)(z-a)(z-b)+\delta(z)(z-a)(z-b)+\epsilon(z)(z-1)(z-b) + \eta(z)(z-1)(z-a)+\frac{(1-\alpha-\beta)(z)(z-1)(z-a)(z-b)}{z-c}$$

$$= \gamma[z^3-z^2(a+b+1)+z(ab+a+b) - ab] + \delta[z^3-(a+b)z^2+abz] +\epsilon[z^3-(b+1)z^2+bz] + \eta[z^3-(a+1)z^2+az] + \frac{(1-\alpha-\beta)(z)(z-1)(z-a)(z-b)}{z-c}.$$

$$(15)\quad K_2 = z^3[\gamma+\delta+\epsilon+\eta]-z^2[\gamma(a+b+1)+\delta(a+b)+\epsilon(b+1)+\eta(a+1)] + z[\gamma(a+b+ab) + \delta(ab) + \epsilon b +\eta a] - ab\gamma + \frac{(1-\alpha-\beta)(z)(z-1)(z-a)(z-b)}{z-c}.$$

We now substitute equation (12) into equation (15), rearrange, and let $c \to \infty$ to obtain:

$$(16)\quad \lim_{c\to\infty} K_2 = z^3(1+\alpha+\beta)-z^2[(1+\alpha+\beta)(1+a+b)-a\epsilon-b\eta-\delta] + z[ab(\gamma+\delta) +a(\gamma+\eta) + b(\gamma+\epsilon)] - ab\gamma.$$

We now let $K_3$ be the coefficient of u, thus:

$$(17)\quad K_3 = \frac{(\alpha\beta)(z)(z-1)(z-a)(z-b)}{(z-c)^2} + D_1(z-1)(z-a)(z-b)$$
$$+ D_2(z)(z-a)(z-b) + D_3(z)(z-1)(z-b)$$
$$+ D_4(z)(z-1)(z-a) + D_5\frac{(z)(z-1)(z-a)(z-b)}{z-c}.$$

Expanding, we get:

$$(18)\quad K_3 = \frac{1}{(z-c)^2}\{\alpha\beta[z^4-z^3(a+b+1) + z^2(a+b+ab) - abz]$$
$$+ D_1[z^5-z^4(a+b+1+2c) + z^3(a+b+ab+2ac+2bc+2c+c^2)$$
$$- z^2(ab+2ac+2bc+2abc+ac^2+bc^2+c^2)$$
$$+ z(ac^2+bc^2+abc^2+2abc) - abc^2]$$
$$+ D_2[z^5-z^4(a+b+2c) + z^3(ab+2ac+2bc+c^2)$$
$$- z^2(2abc+ac^2+bc^2) + abc^2z]$$
$$+ D_3[z^5-z^4(1+b+2c) + z^3(b+2c+2bc+c^2)$$
$$- z^2(2bc+c^2+bc^2) + bc^2z]$$
$$+ D_4[z^5-z^4(a+2c+1) + z^3(a+2ac+2c+c^2)$$
$$- z^2(2ac+ac^2+c^2) + ac^2z]$$
$$+ D_5[z^5-z^4(a+b+c+1) + z^3(a+b+ab+ac+bc+c)$$
$$- z^2(ac+bc+abc+ab) + abcz]\}.$$

Now from equation (18) we isolate coefficients of like powers of z to get the following set of equations:

$$(19)\quad z^5:\ K_4 = \frac{1}{(z-c)^2}(D_1+D_2+D_3+D_4+D_5).$$

(20) $z^4$: $K_5 = \frac{1}{(z-c)^2}[\alpha\beta - D_1(a+b+2c+1) - D_2(a+b+2c)$

$- D_3(b+2c+1) - D_4(a+2c+1)$

$- D_5(a+b+c+1)]$.

(21) $z^3$: $K_6 = \frac{1}{(z-c)^2}[-\alpha\beta(a+b+1) + D_1(a+b+ab+2ac+2bc+2c+c^2)$

$+ D_2(ab+2ac+2bc+c^2)$

$+ D_3(b+2c+2bc+c^2) + D_4(a+2ac+2c+c^2)$

$+ D_5(a+b+ab+ac+bc+c)]$.

(22) $z^2$: $K_7 = \frac{1}{(z-c)^2}[\alpha\beta(a+b+ab) - D_1(ab+2ac+2bc+2abc+ac^2+bc^2+c^2)$

$- D_2(2abc+ac^2+bc^2) - D_3(2bc+c^2+bc^2)$

$- D_4(2ac+ac^2+c^2) - D_5(ac+bc+abc+ab)]$.

(23) $z^1$: $K_8 = \frac{1}{(z-c)^2}[-\alpha\beta ab + D_1(ac^2+bc^2+abc^2+2abc)$

$+ D_2(abc^2) + D_3(bc^2) + D_4(ac^2) + D_5(abc)]$.

(24) $z^0$: $K_9 = \frac{1}{(z-c)^2}[-D_1(abc^2)]$.

We now simplify the above equations.

Substituting equation (9) into equation (19),

(25) we immediately observe that $K_4 = 0$.

Equation (20) can be rearranged as follows:

$$K_5 = \frac{1}{(z-c)^2}\,[\alpha\beta - D_1(a+b+2c+1) - D_2(a+b+2c+1-1)$$
$$- D_3(a+b+2c+1-a) - D_4(a+b+2c+1-b)$$
$$- D_5(a+b+2c+1-c)]$$

$$= \frac{1}{(z-c)^2}\,[-(a+b+2c+1)\,(D_1+D_2+D_3+D_4+D_5)$$
$$+\alpha\beta+D_2+aD_3+bD_4+cD_5].$$

Hence from equations (9) and (10),

(26) $\quad K_5 = 0.$

Simplifying equation (21) slightly, we get:

$$K_6 = \frac{1}{(z-c)^2}\,[-\alpha\beta(a+b+1)+(a+b+ab+2ac+2bc+2c+c^2)$$
$$\cdot(D_1+D_2+D_3+D_4+D_5)$$
$$- D_2(a+b+2c) - D_3(a+ab+2ac)$$
$$- D_4(ab+b+2bc) - D_5(ac+bc+c+c^2)],$$

which by equation (9) reduces to:

$$K_6 = \frac{1}{(z-c)^2}\,[-a\alpha\beta - \alpha\beta(b+1) - D_2(a+b+2c)$$
$$- D_3(a+ab+2ac+a^2-a^2) - D_4(ab+b+2bc)$$
$$- D_5(ac+bc+c+c^2)].$$

$$K_6 = \frac{1}{(z-c)^2}\,[-a(\alpha\beta+D_2+aD_3+bD_4+cD_5) - \alpha\beta(b+1)$$
$$- D_2(b+2c) - D_3(a+ab+2ac-a^2)$$
$$- D_4(b+2bc) - D_5(bc+c+c^2)],$$

which by equation (10) reduces to:

$$K_6 = \frac{1}{(z-c)^2}\,[-\alpha\beta(b+1) - D_2(b+2c) - D_3(a+ab+2ac-a^2)$$
$$- D_4(b+2bc) - D_5(bc+c+c^2)].$$

$$K_6 = \frac{1}{(z-c)^2}\,[-\alpha\beta(2c+b+1-2c) - D_2(b+2c+1-1)$$
$$- D_3(a+ab+2ac-2a^2+a^2) - D_4(b+2bc+b^2-b^2)$$
$$- D_5(bc+c+c^2)]$$
$$= \frac{1}{(z-c)^2}\,[-(2c\alpha\beta+D_2+a^2D_3+b^2D_4+c^2D_5)$$
$$-\alpha\beta(b-2c+1) - D_2(b+2c-1)$$
$$- D_3(a+ab+2ac-2a^2) - D_4(b+2bc-b^2)$$
$$- D_5(bc+c)],$$

which by equation (11) reduces to:

$$K_6 = \frac{1}{(z-c)^2}\,[-\alpha\beta(b+2c-1) - D_2(b+2c-1)$$
$$- D_3(a+ab+2ac-2a^2) - D_4(b+2bc-b^2)$$
$$- D_5(bc+c)].$$

Now letting $c \to \infty$, we get:

(27) $\lim_{c \to \infty} K_6 = 0.$

Equation (22) may be written as follows:

$$K_7 = \frac{1}{(z-c)^2}[\alpha\beta(a+b+ab) - D_1(ab+2ac+2bc+2abc+ac^2+bc^2+c^2)$$
$$- D_2(ab+2ac+2bc+2abc+ac^2+bc^2+c^2-ab-2ac-2bc-c^2)$$
$$- D_3(ab+2ac+2bc+2abc+ac^2+bc^2+c^2-ab-2ac-2abc-ac^2)$$
$$- D_4(ab+2ac+2bc+2abc+ac^2+bc^2+c^2-ab-2bc-2abc-bc^2)$$
$$- D_5(ab+2ac+2bc+2abc+ac^2+bc^2+c^2-ac^2-bc^2-c^2-ac-bc-abc)]$$
$$= \frac{1}{(z-c)^2}[\alpha\beta(a+b+ab)-(ab+2ac+2bc+2abc+ac^2+bc^2+c^2)$$
$$\cdot (D_1+D_2+D_3+D_4+D_5)$$
$$+ D_2(ab+2ac+2bc+c^2) + D_3(ab+2ac+2abc+ac^2)$$
$$+ D_4(ab+2bc+2abc+bc^2) + D_5(ac^2+bc^2+c^2+ac+bc+abc)]$$

which by equation (9) reduces to:

$$K_7 = \frac{1}{(z-c)^2}[\alpha\beta(a+b+ab+2c+c^2-2c-c^2)+D_2(ab+2ac+2bc+c^2)$$
$$+ D_3(ab+2ac+2abc+ac^2)+D_4(ab+2bc+2abc+bc^2)$$
$$+ D_5(ac^2+bc^2+c^2+ac+bc+abc+c^3-c^3)],$$

$$= \frac{1}{(z-c)^2}[c^2(\alpha\beta+D_2+aD_3+bD_4+cD_5)$$
$$+ \alpha\beta(a+b+ab+c^2-2c^2) + D_2(ab+2ac+2bc+c-c)$$
$$+ D_3(a^2c-a^2c+ab+2ac+2abc)$$
$$+ D_4(ab+2bc+2abc+b^2c-b^2c)$$
$$+D_5(ac^2+bc^2+c^2+ac+bc+abc-c^3)].$$

By equation (10), the above expression reduces to:

$$K_7 = \frac{1}{(z-c)^2}[-c(2c\alpha\beta+D_2+a^2D_3+b^2D_4+c^2D_5)+\alpha\beta(a+b+ab+c^2)$$
$$+ D_2(ab+2ac+2bc+c) + D_3(a^2c+ab+2ac+2abc)$$
$$+ D_4(ab+2bc+2abc+b^2c) + D_5(ac^2+bc^2+c^2+ac+bc+abc)].$$

From equation (11), the above equation reduces to:

$$(28)\quad K_7 = \frac{1}{(z-c)^2}[\alpha\beta(a+b+ab+c^2) + D_2(ab+2ac+2bc+c) + D_3(a^2c+ab+2ac+2abc) + D_4(ab+2bc+2abc+b^2c) + D_5(ac^2+bc^2+c^2+ac+bc+abc)].$$

We now eliminate $D_3$ and $D_4$ from equations (9), (10), and (11), to get:

$$(29)\quad D_5 = \frac{(2c-a-b)(\alpha\beta) + abD_1 + (a-1)(b-1)D_2}{(a-c)(c-b)}.$$

Substitute equation (29) into equation (28) to get:

$$K_7 = \frac{1}{(z-c)^2}\{\alpha\beta(a+b+ab+c^2) + D_2(ab+2ac+2bc+c) + D_3(a^2c+ab+2ac+2abc) + D_4(ab+2bc+2abc+b^2c) + [\frac{(2c-a-b)(\alpha\beta)+abD_1+(a-1)(b-1)D_2}{(a-c)(c-b)}][ac^2+bc^2+c^2+ac+bc+abc]\}.$$

Now let $c \to \infty$ to get:

$$(30)\quad \lim_{c\to\infty} K_7 = \alpha\beta.$$

We can rearrange equation (23) slightly as follows:

$$K_8 = \frac{1}{(z-c)^2}[D_1(ac^2+bc^2+abc^2+abc+abc) + D_2(ac^2+bc^2+abc^2+abc-ac^2-bc^2-abc) + D_3(ac^2+bc^2+abc^2+abc-ac^2-abc^2-abc) + D_4(ac^2+bc^2+abc^2+abc-bc^2-abc^2-abc) + D_5(ac^2+bc^2+abc^2+abc-ac^2-bc^2-abc^2) - ab\alpha\beta].$$

$$K_g = \frac{1}{(z-c)^2}\,[(ac^2+bc^2+abc^2+abc)(D_1+D_2+D_3+D_4+D_5)$$
$$+ D_1(abc) - D_2(ac^2+bc^2+abc) - D_3(ac^2+abc^2+abc)$$
$$- D_4(bc^2+abc^2+abc) - D_5(ac^2+bc^2+abc^2) - ab\alpha\beta].$$

By equation (9) this reduces to:

$$K_g = \frac{-1}{(z-c)^2}\,[-D_1(abc) + D_2(ac^2+bc^2+abc+c^2-c^2)$$
$$+ aD_3(ac^2+bc^2+abc+c^2+bc-ac^2-abc)$$
$$+ bD_4(ac^2+bc^2+abc+c^2+ac-bc^2-abc)$$
$$+ cD_5(ac^2+bc^2+abc+c^2+ac+bc-ac^2-bc^2-c^2)$$
$$+ \alpha\beta(ac^2+bc^2+abc+c^2+ab-ac^2-bc^2-abc-c^2)].$$

We rearrange the equation for $K_g$ slightly thus:

$$K_g = \frac{-1}{(z-c)^2}\,[(ac^2+bc^2+abc+c^2)(D_2+aD_3+bD_4+cD_5+\alpha\beta)$$
$$- D_1(abc) - D_2c^2 + aD_3(bc-ac^2-abc)$$
$$+ bD_4(ac-bc^2-abc) + cD_5(ac+bc-ac^2-bc^2-c^2)$$
$$+ \alpha\beta(ab-ac^2-bc^2-abc-c^2)].$$

By equation (10) this reduces to:

$$K_g = \frac{-1}{(z-c)^2}\,[-D_1(abc)+cD_2(a+b-ab-ac-bc-a-b+ab+ac+bc-c)$$
$$+ acD_3(a+b-ab-ac-bc+bc-a)$$
$$+ bcD_4(a+b-ab-ac-bc-b+ac)$$
$$+ c^2D_5(a+b-ab-ac-bc+ab-c)$$
$$+ ab\alpha\beta + c\alpha\beta(a+b-ab-ac-bc-a-b-c)]$$
$$= \frac{-1}{(z-c)^2}[c(\alpha\beta+D_2+aD_3+bD_4+cD_5)(a+b-ab-ac-bc)$$
$$- D_1(abc) + cD_2(ab+ac+bc-a-b-c)$$
$$+ acD_3(bc-a) + bcD_4(ac-b) + c^2D_5(ab-c)$$
$$+ ab\alpha\beta + c\alpha\beta(-a-b-c-c+c)],$$

which by equation (10) reduces to:

$$K_g = \frac{-1}{(z-c)^2}\,[-c(2c\alpha\beta+D_2+a^2D_3+b^2D_4+c^2D_5) - abcD_1$$

$$- c^2D_2 + abc^2D_3 + abc^2D_4 + abc^2D_5$$

$$+ ab\alpha\beta + c\alpha\beta(c-a-b) + cD_2(ab+ac+bc-a-b+1)].$$

By equation (11), this reduces to:

$$K_g = \frac{-1}{(z-c)^2}\,[-abcD_1 + cD_2(ab+ac+bc-a-b-c+1)$$

$$+ abc^2D_3 + abc^2D_4 + abc^2D_5 + ab\alpha\beta$$

$$+ c\alpha\beta(c-a-b)].$$

Letting $c \to \infty$ we get:

$$\lim_{c\to\infty} K_g = -aD_2 - bD_2 + D_2 - abD_3 - abD_4 - abD_5 - \alpha\beta.$$

Now since $D_1 + D_2 = -D_3 - D_4 - D_5$, we write the above expression as follows:

$$\lim_{c\to\infty} K_g = ab(D_1 + D_2) + D_2 - \alpha\beta - D_2(a+b)$$

$$= abD_1 + D_2(ab+1-a-b) - \alpha\beta, \text{ or finally}$$

(31) $$\lim_{c\to\infty} K_g = abD_1 + (a-1)(b-1)D_2 - \alpha\beta.$$

From equation (24), we have immediately,

(32) $$\lim_{c\to\infty} K_9 = -abD_1.$$

Substituting equations (30), (31), and (32) into equation (18) we note that

(33) $$\lim_{c\to\infty} K_3 = \alpha\beta z^2 + [abD_1 + (a-1)(b-1)D_2 - \alpha\beta]z - abD_1.$$

Substituting equations (16), and (33) into equation (13), yields the following differential equation:

$$(34)\quad z(z-1)(z-a)(z-b)\frac{d^2u}{dz^2}+\{z^3(1+\alpha+\beta)-z^2[(1+\alpha+\beta)(1+a+b)-a\epsilon-b\eta-\delta]$$
$$+ z[ab(\gamma+\delta)+a(\gamma+\eta)+b(\gamma+\epsilon)] - ab\gamma\}\frac{du}{dz}$$
$$+ \{\alpha\beta z^2+ z[abD_1+(a-1)(b-1)D_2-\alpha\beta] - abD_1\}\ u = 0.$$

Equation (34) is a second order linear differential equation having $z = 0,1,a,b$, and $\infty$ as regular singular points.

In equation (34) let the coefficient of u be written thus:

$$(35)\quad \alpha\beta z^2+z[abD_1+(a-1)(b-1)D_2-\alpha\beta] - abD_1 \equiv \alpha\beta(z-p)(z-q).$$

Expanding and comparing coefficients of like powers of z we have:

$$(36)\quad -abD_1 = \alpha\beta pq, \text{ and}$$

$$(37)\quad -\alpha\beta(p+q) = [abD_1+(a-1)(b-1)D_2 - \alpha\beta].$$

Solving equations (36) and (37) simultaneously for p and q we get:

$$(38)\quad q = \frac{\alpha\beta-(a-1)(b-1)D_2 - abD_1 \pm R}{2\alpha\beta}, \text{ and}$$

$$(39)\quad p = \frac{2abD_1}{abD_1 + (a-1)(b-1)D_2 - \alpha\beta \pm R}, \text{ where}$$

$$R = \{[abD_1 + (a-1)(b-1)D_2 - \alpha\beta]^2 + 4\alpha\beta abD_1\}^{1/2}.$$

Substituting equation (35) into equation (34) yields:

$$(40)\quad z(z-1)(z-a)(z-b)\frac{d^2u}{dz^2}+\{z^3(1+\alpha+\beta)-z^2[(1+\alpha+\beta)(1+a+b)-a\epsilon-b\eta-\delta]$$

$$+ z[ab(\gamma+\delta)+a(\gamma+\eta)+b(\gamma+\epsilon)] - ab\gamma\}\frac{du}{dz}$$

$$+ \alpha\beta[z^2-z(p+q) + pq]\, u = 0.$$

We may rewrite equation (40) as follows:

$$(41)\quad z(z-1)(z-a)(z-b)\frac{d^2u}{dz^2} + (A_1z^3+A_2z^2+A_3z+A_4)\frac{du}{dz}$$

$$+ (B_1z^2+B_2z+B_3)\, u = 0, \text{ where}$$

$$(42)\quad A_1 = (1+\alpha+\beta),$$

$$(43)\quad A_2 = -[(1+\alpha+\beta)(1+a+b) -a\epsilon-b\eta-\delta],$$

$$(44)\quad A_3 = [ab(\gamma+\delta)+a(\gamma+\eta)+b(\gamma+\epsilon)],$$

$$(45)\quad A_4 = -ab\gamma,$$

$$(46)\quad B_1 = \alpha\beta,$$

$$(47)\quad B_2 = -\alpha\beta(p+q), \text{ and}$$

$$(48)\quad B_3 = \alpha\beta pq, \text{ where } q \text{ and } p \text{ are given by equations (38) and (39) above.}$$

Equation (41) is the differential equation which we wish to solve.

## CHAPTER VI

## A GENERAL SOLUTION OF A GENERAL DIFFERENTIAL EQUATION WITH FIVE REGULAR SINGULAR POINTS

We now consider a solution of the differential equation:

$$(1)\qquad z(z-1)(z-a)(z-b)\frac{d^2u}{dz^2} + (A_1z^3+A_2z^2+A_3z+A_4)\frac{du}{dz} + (B_1z^2+B_2z+B_3)u = 0,$$

where

$$(2)\qquad A_1 = (1 + \alpha + \beta),$$

$$(3)\qquad A_2 = -[(1 + \alpha + \beta)(1 + a + b) - a\epsilon - b\eta - \delta],$$

$$(4)\qquad A_3 = [ab(\gamma + \delta) + a(\gamma + \eta) + b(\gamma + \epsilon)],$$

$$(5)\qquad A_4 = -ab\gamma,$$

$$(6)\qquad B_1 = \alpha\beta,$$

$$(7)\qquad B_2 = -\alpha\beta(p + q), \text{ and}$$

$$(8)\qquad B_3 = \alpha\beta pq.$$

We will initially obtain a solution of the differential equation in a neighborhood of $z = 0$.

$$(9)\qquad \text{Let } u(z) = c_0z^{\lambda} + c_1z^{\lambda+1} + \cdots + c_{r-2}z^{\lambda+r-2} + c_{r-1}z^{\lambda+r-1} + c_rz^{\lambda+r} + \cdots ;\ c_0 \neq 0$$

be a solution of equation (1), then

$$(10)\qquad u'(z) = c_0\lambda z^{\lambda-1} + c_1(\lambda+1)z^{\lambda} + \cdots + (c_{r-2})(\lambda+r-2)z^{\lambda+r-3} + (c_{r-1})(\lambda+r-1)z^{\lambda+r-2} + (c_r)(\lambda+r)z^{\lambda+r-1} + \cdots , \text{ and}$$

(11) $u''(z) = c_0\lambda(\lambda-1)z^{\lambda-2} + c_1(\lambda+1)(\lambda)z^{\lambda-1} + \cdots$

$+ (c_{r-2})(\lambda+r-2)(\lambda+r-3)z^{\lambda+r-4}$

$+ (c_{r-1})(\lambda+r-1)(\lambda+r-2)z^{\lambda+r-3}$

$+ (c_r)(\lambda+r)(\lambda+r-1)z^{\lambda+r-1} + \cdots$ .

Substituting equations (9), (10), and (11) into equation (1) and equating the sum of the coefficients of the lowest power of z to zero, we get:

(12) $z^{\lambda-1}$: $-abc_0\lambda(\lambda - 1) + A_4c_0\lambda = 0$, but from equation (5), $A_4 = -ab\gamma$, hence equation (12) becomes:

$-abc_0\lambda(\lambda-1) - ab\gamma c_0\lambda = 0$. Now $ab \neq 0$, since a and b are distinct non-zero regular singular points, also $c_0 \neq 0$ by assumption, hence equation (12) becomes:

(13) $\lambda(\lambda-1) + \lambda\gamma = 0$, whence

$\lambda = 0$, or $\lambda = 1 - \gamma$ . This is a verification that the exponents at $z = 0$ are 0 and $1 - \gamma$ as chosen. Now

$z(z-1)(z-a)(z-b) \equiv z^4 - (a+b+1)z^3 + (a+b+ab)z^2 - abz$,

thus we can compute the sum of the coefficients of $z^{\lambda+r}$:

(14) $[(c_{r-2})(\lambda+r-2)(\lambda+r-3) - (a+b+1)(c_{r-1})(\lambda+r-1)(\lambda+r-2)$

$+ (a+b+ab)(c_r)(\lambda+r)(\lambda+r-1)-(ab)(c_{r+1})(\lambda+r+1)(\lambda+r)]$

$+ [(A_1)(c_{r-2})(\lambda+r-2)+(A_2)(c_{r-1})(\lambda+r-1)+(A_3)(c_r)(\lambda+r)$

$+(A_4)(c_{r+1})(\lambda+r+1)]+[(B_1)(c_{r-2})+(B_2)(c_{r-1})+(B_3)(c_r)] = 0.$

Setting $\lambda = 0$, and simplifying equation (14) somewhat, we get:

$$(15)\quad c_{r+1}[(ab)(r+1)(r) - A_4(r+1)]$$

$$= c_r[(a+b+ab)(r)(r-1) + rA_3 + B_3]$$

$$+ c_{r-1}[A_2(r-1) + B_2 - (a+b+1)(r-1)(r-2)]$$

$$+ c_{r-2}[(r-2)(r-3) + A_1(r-2) + B_1].$$

Equation (15) is a four term recurrence relationship for generating $c_{r+1}$, when we assume a solution of equation (1) to be

$$u(z) = \sum_{r=0}^{\infty} c_r z^{\lambda+r}; \quad c_o \neq 0.$$

In equation (15) let $r = 0$, then

$$c_1(-A_4) = B_3 c_o; \quad c_{-1} = c_{-2} = 0.$$

$$c_1 = \frac{-B_3 c_o}{A_4} = \frac{-\alpha\beta pq c_o}{-ab}, \text{ hence}$$

$$c_1 = \frac{\alpha\beta pq c_o}{ab\gamma} \text{ from equations (5) and (8).}$$

Now in equation (15) let $r = 1$ to get:

$c_2[2ab - 2A_4] = c_1[A_3 + B_3] + c_o[B_2]$, which upon substituting equations (4), (5), (7), and (8) reduces to

$$c_2[2ab + 2ab\gamma] = c_1[ab(\gamma+\delta) + a(\gamma+\eta) + b(\gamma+\epsilon) + \alpha\beta pq]$$

$$-\alpha\beta(p+q)c_o. \quad \text{But } c_1 = \frac{\alpha\beta pq c_o}{ab\gamma}, \text{ hence}$$

$$(17)\qquad c_2 = \frac{\{pq[ab(\gamma+\delta)+a(\gamma+\eta)+b(\gamma+\epsilon)+\alpha\beta pq]-ab\gamma(p+q)\}\{\alpha\beta c_o\}}{2!(ab)^2(\gamma)(\gamma+1)}.$$

We could continue to compute constants $c_{r+1}$ from the recurrence equation (15); however, we choose to define a general solution of equation (1) in terms of another set of constants $G_r(p,q)$, after first demonstrating the connection between $c_r$ and $G_r(p,q)$.

Let $G_r \equiv G_r(p,q)$, $(r = 0,1,\ldots)$, and define

$G_o = c_o$,

$G_1 = \alpha\beta pqc_o$, and

$G_2 = \alpha\beta c_o\{pq[ab(\gamma+\delta)+a(\gamma+\eta)+b(\gamma+\epsilon)+\alpha\beta pq]-ab\gamma(p+q)\}$.

Since we are expanding in a neighborhood of $z = 0$, we may either write a solution of equation (1) in the form:

$$(18)\qquad u(z) = c_o + \sum_{r=1}^{\infty} c_r z^r, \text{ where } \lambda = 0, \text{ or}$$

$$(19)\qquad u(z) = G_o + \sum_{r=1}^{\infty} \frac{G_r z^r}{r!(ab)^r\gamma(\gamma+1)\cdots(\gamma+r-1)}.$$

Comparing equations (18) and (19), we observe that

$$(20)\qquad c_r = \frac{G_r}{r!(ab)^r(\gamma)(\gamma+1)\cdots(\gamma+r-1)}, \quad (r = 1,2,\ldots).$$

Substituting equation (20) into equation (15) yields:

$$(21)\qquad \frac{G_{r+1}[(ab)(r+1)(r) - A_4(r+1)]}{(r+1)!\,(ab)^{r+1}(\gamma)(\gamma+1)\cdots(\gamma+r)}$$

$$= \frac{G_r[(a+b+ab)(r)(r-1) + A_3 r + B_3]}{r!(ab)^r(\gamma)(\gamma+1)\cdots(\gamma+r-1)}$$

$$+ \frac{G_{r-1}[A_2(r-1)+B_2-(a+b+1)(r-1)(r-2)}{(r-1)!(ab)^{r-1}(\gamma)(\gamma+1)\cdots(\gamma+r-2)}$$

$$+ \frac{G_{r-2}[(r-2)(r-3)+A_1(r-2)+B_1]}{(r-2)!(ab)^{r-2}(\gamma)(\gamma+1)\cdots(\gamma+r-3)}, \text{ where}$$

$A_i, (i = 1,2,3,4$ is given by equation (2) - (5), and

$B_j, (j = 1,2,3$ is given by equation (6) - (8).

Substituting equation (5) into equation (21) we get:

(22)
$$\frac{G_{r+1}[(ab)(r+1)(r+\gamma)]}{(r+1)!(ab)^{r+1}(\gamma)(\gamma+1)\cdots(\gamma+r)}$$

$$= \frac{G_r[(a+b+ab)(r)(r-1)+A_3r+B_3]}{r!(ab)^r(\gamma)(\gamma+1)\cdots(\gamma+r-1)}$$

$$+ \frac{G_{r-1}[A_2(r-1)+B_2-(a+b+1)(r-1)(r-2)]}{(r-1)!(ab)^{r-1}(\gamma)(\gamma+1)\cdots(\gamma+r-2)}$$

$$+ \frac{G_{r-2}[(r-2)(r-3)+A_1(r-2)+B_1]}{(r-2)!(ab)^{r-2}(\gamma)(\gamma+1)\cdots(\gamma+r-3)}.$$

We now divide numerator and denominator of the left side of equation (22) by $(ab)(r+1)(r+\gamma)$, then multiply both sides of the result by $r!(ab)^r\cdot(\gamma)(\gamma+1)\cdots(\gamma+r-1)$ to get:

(23)
$$G_{r+1} = G_r[(a+b+ab)(r)(r-1)+A_3r+B_3]$$

$$+ G_{r-1}[A_2(r-1)+B_2-(a+b+1)(r-1)(r-2)](r)(ab)(\gamma+r-1)$$

$$+ G_{r-2}[(r-2)(r-3)+A_1(r-2)+B_1](r)(r-1)(ab)^2(\gamma+r-1)(\gamma+r-2).$$

Now substituting equations (2) - (4) and (6) - (8) into equation (23), and remembering that $G_r \equiv G_r(p,q)$, we get:

(24) $G_{r+1}(p,q) =$

$G_r(p,q)\{(a+b+ab)(r)(r-1)+r[ab(\gamma+\delta)+a(\gamma+\eta)+b(\gamma+\epsilon)]+\alpha\beta pq\}$

$- G_{r-1}(p,q)\{(a+b+1)(r-1)(r-2)+(r-1)[(1+\alpha+\beta)(1+a+b)-a\epsilon-b\eta-\delta]$

$+ \alpha\beta(p+q)\}[(r)(ab)(\gamma+r-1)]+G_{r-2}(p,q)\{(r-2)(r-3)$

$+ (r-2)(1+\alpha+\beta)+\alpha\beta\}[(r)(r-1)(ab)^2(\gamma+r-1)(\gamma+r-2)].$

This is a four term recurrence equation which can be used to generate coefficients $G_r(p,q)$; $r = 1,2,\cdots$. These coefficients can then be substituted into equation (19) to give a general solution of equation (1) in a neighborhood of $z = 0$.

Equation (24) is a difference equation which is undoubtedly more difficult to solve than the original differential equation. Consequently we shall establish criteria under which there exists a two term recurrence formula connecting the coefficients in the solution of a second order linear differential equation with five regular singular points. We shall then apply these criteria to equation (1), and hence get solutions in a neighborhood of each of the points $z = 0$, $1,a,b,\infty$, and d, where $d \neq 0,1,a,b$, or $\infty$.

The aforementioned criteria were first established by Scheffé [13, pp. 240-49].

# CHAPTER VII

## ON SCHEFFÉ'S CRITERIA

The solution of equation (1) in Chapter VI involved the use of a complicated four term recurrence equation.

It is our purpose in Chapter VII to establish criteria under which a solution of a linear ordinary second order differential equation will have a two term recurrence relationship between successive coefficients in its series expansion. Criteria for such a recurrence relation were established by Scheffé [13, p. 240 ] for an n-th order linear ordinary differential equation.

Consider the following differential equation:

(1) $$p_2(z)u''(z) + p_1(z)u'(z) + p_0(z)u(z) = 0.$$

We assume that $p_j(z)$ are analytic and $p_j(z) \not\equiv 0$, $(j = 0,1,2)$, in a region R. We further assume a solution of equation (1) in a neighborhood of $z = z_o$ to be of the form:

(2) $$u(z) = \sum_{r=0}^{\infty} c_r(z-z_o)^{\lambda+r}$$, where $\lambda = \lambda_1, \lambda_2$ are roots of the indicial equation, and $c_o \neq 0$.

We now define the region R as consisting of all points in an arbitrary neighborhood of $z = z_o$, except the point $z_o$.

Multiplying both members of equation (1) by $(z-z_o)^m$, where m is an integer, we get:

(3) $$p_2(z)(z-z_o)^m u''(z) + p_1(z)(z-z_o)^m u'(z) + p_0(z)(z-z_o)^m u(z) = 0.$$

We now define a new function $q_j(z)$ as follows:

(4) $q_j(z) = p_j(z)(z-z_o)^{m-j}$, $(j = 0,1,2)$, and substitute equation (4) into equation (3) thus obtaining:

(5) $$q_2(z)(z-z_o)^2 u''(z) + q_1(z)(z-z_o)u'(z) + q_0(z)u(z) = 0.$$

We now expand $q_j(z)$, $(j = 0,1,2)$, in a Taylor's series about $z = z_o$ to get:

(6) $q_j(z) = \sum_{k=0}^{\infty} q_{j,k}(z-z_o)^k$, or upon expansion,

equation (6) gives us:

(7) $$q_0(z) = q_{0,0} + q_{0,1}(z-z_o) + \cdots + q_{0,k}(z-z_o)^k + \cdots ,$$

(8) $$q_1(z) = q_{1,0} + q_{1,1}(z-z_o) + \cdots + q_{1,k}(z-z_o)^k + \cdots ,$$

and

(9) $$q_2(z) = q_{2,0} + q_{2,1}(z-z_o) + \cdots + q_{2,k}(z-z_o)^k + \cdots ,$$

where $q_{j,k}$, $(j = 0,1,2)$ and $(k = 0,1,2,\cdots)$, are constants to be determined.

We will now demonstrate the existence of an integer m such that not all of $q_j(z_o)$ are zero.

Let $p_j(z)$, $(j = 0,1,2)$, be expanded in a Taylor's series of the form:

$$(10) \quad p_0(z) = A_t(z-z_o)^t + A_{t+1}(z-z_o)^{t+1} + \cdots + A_{t+i}(z-z_o)^{t+i} + \cdots ,$$

$$(11) \quad p_1(z) = B_s(z-z_o)^s + B_{s+1}(z-z_o)^{s+1} + \cdots + B_{s+i}(z-z_o)^{s+i} + \cdots , \text{ and}$$

$$(12) \quad p_2(z) = c_n(z-z_o)^n + c_{n+1}(z-z_o)^{n+1} + \cdots + c_{n+i}(z-z_o)^{n+i} + \cdots ,$$

where t, s, and n are non-negative integers, and $(A_t)(B_s)(C_n) \neq 0$. We note that if $(t)(s)(n) > 0$, then $p_j(z_o) = 0$, contrary to the restriction $p_j(z) \not\equiv 0$.

Combining equation (4) and equations (10) - (12) we have:

$$(13) \qquad q_0(z) = A_t(z-z_o)^{m+t} + A_{t+1}(z-z_o)^{m+t+1} + \cdots + A_{t+i}(z-z_o)^{m+t+i} + \cdots,$$

$$(14) \qquad q_1(z) = B_s(z-z_o)^{m+s-1} + B_{s+1}(z-z_o)^{m+s} + \cdots + B_{s+i}(z-z_o)^{m+s-i+1} + \cdots, \text{ and}$$

$$(15) \qquad q_2(z) = C_n(z-z_o)^{m+n-2} + C_{n+1}(z-z_o)^{m+n-1} + \cdots + C_{n+i}(z-z_o)^{m+n-i+2} + \cdots.$$

Now $m+t \geq 0$, $m+s-1 \geq 0$, and $m+n-2 \geq 0$, for otherwise $q_j(z)$ may fail to be analytic, contrary to equation (6) which implies analyticity of $q_j(z)$ not only at $z = z_o$, but also at all other points in the region R.

Suppose $m+t = m+s-1 = m+n-2 = 0$. Then from equations (13)-(15), $(A_t)(B_s)(C_n) \neq 0$, which implies from equations (7)-(9) that $(q_{0,0})(q_{1,0})(q_{2,0}) \neq 0$, and the condition that not all $q_j(z_o)$ be zero is fulfilled.

We next suppose $m+t \leq m+n-2$, $m+t \leq m+s-1$, and choose $m = -t$. Then from equations (7) and (13) we note that $q_{0,0} = A_t \neq 0$, since $(A_t)(B_s)(C_n) \neq 0$.

Again we suppose $m+s-1 \leq m+t$, and $m+s-1 \leq m+n-2$, and we choose $m = 1-s$. Comparison of equations (8) and (14) reveals that $q_{1,0} = B_s \neq 0$.

Finally we suppose $m+n-2 \leq m+t$, and $m+n-2 \leq m+s-1$, and choose $m = 2-n$. Comparison of equations (9) and (15) reveals that for this choice of m, we have $q_{2,0} = C_n \neq 0$.

This completes the demonstration of the existence of an integer m such that at least one of the numbers $m+t$, $m+s-1$ and $m+n-2$ can be made to vanish, which implies m can be chosen such that not all of $q_j(z_o) = 0$, $(j = 0,1,2)$.

Writing equation (2) in expanded form we have:

$$(16) \qquad u(z) = c_o(z-z_o)^{\lambda} + c_1(z-z_o)^{\lambda+1} + \cdots + c_{r-1}(z-z_o)^{\lambda+r-1} + c_r(z-z_o)^{\lambda+r} + \cdots .$$

Differentiating equation (16) with respect to $(z-z_o)$ yields:

$$(17) \qquad u'(z) = c_o\lambda(z-z_o)^{\lambda-1} + c_1(\lambda+1)(z-z_o)^{\lambda} + \cdots + c_{r-1}(\lambda+r-1)(z-z_o)^{\lambda+r-2} + c_r(\lambda+r)(z-z_o)^{\lambda+r-1} + \cdots ,$$

where it is to be understood that

$$u'(z) = \frac{du(z)}{d(z-z_o)} \equiv \frac{du(z)}{dz} .$$

Differentiating equation (17) with respect to $(z-z_o)$ yields:

$$(18)\qquad u''(z) = c_0\lambda(\lambda-1)(z-z_0)^{\lambda-2} + c_1(\lambda+1)(\lambda)(z-z_0)^{\lambda-1} + \cdots$$
$$+ c_{r-1}(\lambda+r-1)(\lambda+r-2)(z-z_0)^{\lambda+r-3}$$
$$+ c_r(\lambda+r)(\lambda+r-1)(z-z_0)^{\lambda+r-2} + \cdots .$$

We now multiply both members of $q_j(z)$ as given by equations(7)-(9), by $(z-z_0)^j$, $(j = 0,1,2)$, then substitute the results into equation (5) to get:

$$(19)\qquad [q_{2,0}(z-z_0)^2 + q_{2,1}(z-z_0)^3 + \cdots + q_{2,k}(z-z_0)^{k+2}$$
$$+ q_{2,k+1}(z-z_0)^{k+3} + \cdots]u''(z)$$
$$+ [q_{1,0}(z-z_0) + q_{1,1}(z-z_0)^2 + \cdots + q_{1,k}(z-z_0)^{k+1}$$
$$+ q_{1,k+1}(z-z_0)^{k+2} + \cdots]u'(z)$$
$$+ [q_{0,0} + q_{0,1}(z-z_0) + \cdots + q_{0,k}(z-z_0)^k$$
$$+ q_{0,k+1}(z-z_0)^{k+1} + \cdots]u(z) = 0,$$

where $u''(z)$, $u'(z)$, and $u(z)$ are given by equations (16) - (18).

Equating the coefficient of $(z-z_0)^\lambda$ to zero, we obtain the indicial equation, thus:

$$(20)\qquad q_{2,0}c_0(\lambda)(\lambda-1) + q_{1,0}c_0\lambda + q_{0,0}c_0 = 0,$$

where $c_0 \neq 0$, hence

$$(21)\qquad \lambda(\lambda-1)q_{2,0} + \lambda q_{1,0} + q_{0,0} = 0.$$

Now in Pochhammer's notation, [11],

$[\lambda]_j = (\lambda)(\lambda-1)\cdots(\lambda-j+1)$, if $j$ is a positive integer, and $[\lambda]_0 = 1$ by definition. Thus equation (21), when expressed in Pochhammer's notation, becomes:

$$(22) \qquad q_{2,0}[\lambda]_2 + q_{1,0}[\lambda]_1 + q_{0,0}[\lambda]_0 = 0.$$

Similarly the coefficient of $(z-z_o)^{\lambda+1}$ is

$$(23) \qquad c_1[(\lambda+1)(\lambda)q_{2,0} + (\lambda+1)q_{1,0} + q_{0,0}] + c_o[\lambda(\lambda-1)q_{2,1} + \lambda q_{1,1} + q_{0,1}] = 0, \text{ or in}$$

Pochhammer's notation, equation (23) becomes:

$$(24) \qquad c_1\{q_{0,0}[\lambda+1]_0 + q_{1,0}[\lambda+1]_1 + q_{2,0}[\lambda+1]_2\} + c_o\{q_{0,1}[\lambda]_0 + q_{1,1}[\lambda]_1 + q_{2,1}[\lambda]_2\} = 0.$$

We now write the coefficient of the general term, namely $(z-z_o)^{\lambda+r}$:

$$(25)\quad \{c_r[q_{2,0}(\lambda+r)(\lambda+r-1) + q_{1,0}(\lambda+r) + q_{0,0}]$$

$$+ c_{r-1}[q_{2,1}(\lambda+r-1)(\lambda+r-2) + q_{1,1}(\lambda+r-1) + q_{0,1}]$$

$$+ c_{r-2}[q_{2,2}(\lambda+r-2)(\lambda+r-3) + q_{1,2}(\lambda+r-2) + q_{0,2}]$$

$$+ \cdot$$
$$\cdot$$
$$\cdot$$

$$+ c_{r-m}[q_{2,m}(\lambda+r-m)(\lambda+r-m-1) + q_{1,m}(\lambda+r-m) + q_{0,m}]$$

$$+ \cdot$$
$$\cdot$$
$$\cdot$$

$$+ c_o[q_{2,r}(\lambda)(\lambda-1) + \lambda q_{1,r} + q_{0,r}]\} = 0.$$

Equation (25), using Pochhammer's notation, becomes:

$$(26)\quad \{c_r(q_{2,0}[\lambda+r]_2 + q_{1,0}[\lambda+r]_1 + q_{0,0}[\lambda+r]_0)$$

$$+ c_{r-1}(q_{2,1}[\lambda+r-1]_2 + q_{1,1}[\lambda+r-1]_1 + q_{0,1}[\lambda+r-1]_0)$$

$$+ c_{r-2}(q_{2,2}[\lambda+r-2]_2 + q_{1,2}[\lambda+r-2]_1 + q_{0,2}[\lambda+r-2]_0)$$

$$+ \cdot$$
$$\cdot$$
$$\cdot$$

$$+ c_{r-m}(q_{2,m}[\lambda+r-m]_2 + q_{1,m}[\lambda+r-m]_1 + q_{0,m}[\lambda+r-m]_0)$$

$$+ \cdot$$
$$\cdot$$
$$\cdot$$

$$+ c_o(q_{2,r}[\lambda]_2 + q_{1,r}[\lambda]_1 + q_{0,r}[\lambda]_0)\} = 0.$$

We now define

(27) $Q_m(x) = \sum_{t=0}^{2} q_{t,m}[x]_t$, so that equation (26) may be written as follows:

(28) $\sum_{m=0}^{r} Q_m(\lambda+r-m)c_{r-m} = 0$, or in expanded form, equation (28) becomes:

(29) $$Q_0(\lambda+r)c_r + Q_1(\lambda+r-1)c_{r-1} + \cdots + Q_m(\lambda+r-m)c_{r-m} + \cdots + Q_r(\lambda)c_0 = 0.$$

For m = h, some positive integer, we note that

(30) $$Q_h(\lambda+r-h) = q_{2,h}[\lambda+r-h]_2 + q_{1,h}[\lambda+r-h]_1 + q_{0,h}[\lambda+r-h]_0,$$

also when m = 0, we have:

(31) $$Q_0(\lambda+r) = q_{2,0}[\lambda+r]_2 + q_{1,0}[\lambda+r]_1 + q_{0,0}[\lambda+r]_0 \not\equiv 0,$$

since by hypothesis not all of $q_{0,0}$, $q_{1,0}$ and $q_{2,0}$ vanish simultaneously.

Now by definition, equation (1) is said to have a two term recurrence equation relative to $z = z_0$ if the recurrence equation is of the form:

(32) $f(r,\lambda)c_r + g(r,\lambda)c_{r-h} = 0$, where $r = h, h+1, \cdots$, and where h is some positive integer. Also, neither $f(r,\lambda)$ nor $g(r,\lambda)$ is identically zero in both r and $\lambda$, otherwise equation (32) would not contain two non-zero terms.

Inasmuch as $Q_0(\lambda+r) \not\equiv 0$ from equation (31), we note that necessary and sufficient conditions for equation (32) to be a two term recurrence equation are that

(33) $Q_h(\lambda+r-h) \not\equiv 0$, and

(34) $Q_m(\lambda+r-m) = 0$, $(m \neq 0,h)$.

The necessity of equation (33) is evident. Equation (34) may be analyzed as follows: The $Q_m(\lambda+r-m)$ are polynomials of degree less than or equal to 2, and can vanish for $r-m = h,h+1, h+2$, and fixed $\lambda$, only if $Q_m(\lambda+r-m) = 0$. Hence equations (33) and (34) are equivalent to

(35) $q_{j,h} \not\equiv 0$ for some $(j = 0,1,2)$, and

(36) $q_{j,m} = 0$ for $(j = 0,1,2;\ m \neq 0,h)$.

The coefficients of $p_j(z)$ in equation (9) are then of the form:

(37) $p_j(z) = [q_{j,0} + q_{j,h}(z-z_o)^h][z-z_o]^{j-m}$.

The above results are summarized in the following theorem:

Theorem: Necessary and sufficient conditions for a solution of a second order linear ordinary differential equation to have a two term recurrence equation, relative to a point $z = z_o$, is that in some neighborhood of the point $z = z_o$,

(38) $p_j(z) = [S_j - T_j(z-z_o)^h][z-z_o]^{j-m}$, where m is an integer, h is a positive integer and the constants $S_j = q_{j,0}$, $T_j = -q_{j,h}$ are such that $S_i T_j \neq 0$ for some $(i = 0,1,2)$ and $(j = 0,1,2)$.

In Chapter VIII we will apply the above criteria to a second order linear ordinary differential equation with five regular singular points.

## CHAPTER VIII

## ON THE APPLICATION OF SCHEFFÉ'S CRITERIA TO THE EQUATION WITH FIVE REGULAR SINGULAR POINTS

### Solution in a Neighborhood of z = 0

The differential equation to which we will apply Scheffé's criteria was derived in Chapter V, and is repeated here for convenience:

$$(1)\qquad (z)(z-1)(z-a)(z-b)\frac{d^2u}{dz^2} + (A_1z^3 + A_2z^2 + A_3z + A_4)\frac{du}{dz} + (B_1z^2 + B_2z + B_3)u = 0, \text{ where}$$

$$(2)\qquad A_1 = (1+\alpha+\beta),$$

$$(3)\qquad A_2 = -[(1+\alpha+\beta)(1+a+b) - a\epsilon - b\eta - \delta],$$

$$(4)\qquad A_3 = [ab(\gamma+\delta) + a(\gamma+\eta) + b(\gamma+\epsilon)],$$

$$(5)\qquad A_4 = -ab\gamma,$$

$$(6)\qquad B_1 = \alpha\beta,$$

$$(7)\qquad B_2 = -\alpha\beta(p+q), \text{ and}$$

$$(8)\qquad B_3 = \alpha\beta pq.$$

We may write equation (1) as follows:

$$(9)\qquad p_2(z)\frac{d^2u}{dz^2} + p_1(z)\frac{du}{dz} + p_o(z)u = 0, \text{ where}$$

$$(10)\qquad p_2(z) = (z)(z-1)(z-a)(z-b),$$

$$(11)\qquad p_1(z) = A_1z^3 + A_2z^2 + A_3z + A_4, \text{ and}$$

$$(12)\qquad p_o(z) = B_1z^2 + B_2z + B_3.$$

Scheffé's criteria, which were derived in Chapter VII, proves that $p_j(z)$ is written

(13) $\quad p_j(z) = [S_j - T_j(z-z_o)^h][z-z_o]^{j-m}$, $(j = 0,1,2)$,

whenever a solution to equation (1), in a neighborhood of $z = z_o$, has a two term recurrence equation. We recall the fundamental criteria in connection with equation (13) such as:

(14) $\quad S_iT_j \neq 0$ for some $(i = 0,1,2)$ and $(j = 0,1,2)$, h is a positive integer, and m is an integer.

If we assume a solution to equation (1) in a neighborhood of $z = z_o$, we initially make the following variable change

(15) $\quad w = z - z_o$, or

(16) $\quad z = w + z_o$. We then expand using the method of Frobenius [3, p. 214] in a neighborhood of $w = 0$ as follows:

(17) $\quad u(w) = \sum_{r=0}^{\infty} c_{3r}w^{\lambda+3r}$; $c_o \neq 0$, or in expanded form, equation (17) becomes:

(18) $\quad u(w) = c_ow^{\lambda} + c_3w^{\lambda+3} + \cdots + c_{3r}w^{\lambda+3r}$

$$+ c_{3r+3}w^{\lambda+3r+3} + \cdots .$$

Differentiating equation (18) with respect to w yields:

$$(19) \qquad u'(w) = c_0\lambda w^{\lambda-1} + c_3(\lambda+3)w^{\lambda+2} + \cdots + c_{3r}(\lambda+3r)w^{\lambda+3r-1} + c_{3r+3}(\lambda+3r+3)w^{\lambda+3r+2} + \cdots ,$$

where $u'(w) = \frac{du}{dw}$.

Differentiating equation (19) with respect to w, gives the following expression:

$$(20) \qquad u''(w) = c_0\lambda(\lambda-1)w^{\lambda-2} + c_3(\lambda+3)(\lambda+2)w^{\lambda+1} + \cdots + c_{3r}(\lambda+3r)(\lambda+3r-1)w^{\lambda+3r-2} + c_{3r+3}(\lambda+3r+3)(\lambda+3r+2)w^{\lambda+3r+1} + \cdots .$$

The preceding relationships, together with related theory, will serve as a basis on which we can proceed to apply Scheffé's criteria to equation (1).

Since we are initially expanding in a neighborhood of $z = 0$, we let $z_0 = 0$ in the preceding equations, and let $w \equiv z$. Thus there is no change in equation (1).

In equation (13) we let $j = 2$, and assume $S_2 = 0$. We now compare equations (10) and (13) and deduce the following identity when $z_0 = 0$:

$$(21) \qquad -T_2z^{h-m+2} \equiv z^4 - z^3(a+b+1) + z^2(a+ab+b) - abz.$$

Evidently $h - m + 2 = 4$, otherwise the coefficient of $z^4$ in the left member of identity (21) is zero, while

the coefficient of $z^4$ in the right member is unity. This is an obvious inconsistency.

Equating like powers of z in identity (21) we get the following set of equations:

(i) $T_2 = -1$,

(ii) $a+b+1 = 0$,

(iii) $a+ab+b = 0$, and

(iv) $ab = 0$.

An attempt to solve the above equations leads to an inconsistency, hence we conclude $S_2 \neq 0$.

We next assume $T_2 = 0$, and again comparing equations (10) and (13) yields the following identity:

(22) $S_2 z^{2-m} \equiv z^4 - z^3(a+b+1) + z^2(a+ab+b) - abz.$

As before, $2 - m = 4$, otherwise $S_2 = 0$ and we have the inconsistency previously mentioned.

Equating the coefficients of like powers of z in identity (22) gives the preceding set of inconsistent equations. We therefore conclude $S_2 \neq 0$. We now have $S_2 T_2 \neq 0$, and restriction (14) is satisfied.

Since neither $S_2$ nor $T_2$ is zero, we again compare equations (10) and (13) to get the following identity:

(23) $$p_2(z) = S_2 z^{2-m} - T_2 z^{h-m+2}$$
$$\equiv z^4 - z^3(a+b+1) + z^2(a+b+ab) - abz.$$

Now since h is a positive integer, $h - m + 2 > 2 - m$, therefore $h - m + 2 = 4$, or

(24) $h - m = 2$.

From equation (24) we deduce the following set of values for h and m:

Case I: $h = 1$; $m = -1$,

Case II: $h = 2$; $m = 0$, and

Case III: $h = 3$; $m = 1$.

That this set of cases is exhaustive may be demonstrated as follows: From Scheffé's criteria, h is a positive integer. Thus m cannot be less than -1. Suppose $h \geq 4$, then from equation (24) $m \geq 2$. But from identity (23), $m \geq 2$ implies $S_2 = 0$, which is impossible. Therefore the above cases form an exhaustive set.

We will investigate each of these cases in turn.

Case I: h = 1; m = -1

For these values of h and m, identity (23) becomes:

$$(25) \qquad p_2(z) = S_2z^3 - T_2z^4 \equiv z^4 - z^3(a+b+1) + z^2(a+b+ab) - abz.$$

Equating coefficients of like powers of z in this identity gives us the following set of equations:

(i) $T_2 = -1$,

(ii) $S_2 = -(a+b+1)$,

(iii) $a+b+ab = 0$, and

(iv) $ab = 0$.

From this set of equations we conclude that $S_2 = -1$, and equation (13) becomes:

$$(26) \qquad p_2(z) = z^4 - z^3.$$

Comparing equations (11) and (13), where j = 1, h = 1 and m = -1, gives the following expression:

$$(27) \qquad p_1(z) = S_1z^2 - T_1z^3 \equiv A_1z^3 + A_2z^2 + A_3z + A_4.$$

Proceeding as before, we get the following set of equations:

(i) $T_1 = -A_1 = -(1+\alpha+\beta)$,

(ii) $S_1 = A_2 = -[(1+\alpha+\beta)(1+a+b) - a\epsilon - b\eta - \delta]$,

(iii) $A_3 = 0 = [ab(\gamma+\delta) + a(\gamma+\eta) + b(\gamma+\epsilon)]$, and

(iv) $A_4 = 0 = ab\gamma$.

There are several choices at our disposal from equation (iv); however, we choose $a = b = 0$, $\gamma \neq 0$. Equation (27) then simplifies to:

$$(28) \qquad p_1(z) = (1+\alpha+\beta)z^3 - (1+\alpha+\beta-\delta)z^2.$$

Now comparing equations (12) and (13) with $j = 0$, $h = 1$, and $m = -1$, we have the following identity:

(29) $p_0(z) = S_0 z - T_0 z^2 = B_1 z^2 + B_2 z + B_3$, from which we get the following set of equations:

(i) $T_0 = -B_1 = -\alpha\beta$,

(ii) $S_0 = B_2 = -\alpha\beta(p+q)$, and

(iii) $B_3 = 0 = \alpha\beta pq$.

Now from equation (iii), $\alpha\beta \neq 0$, otherwise equation (1) becomes essentially a first order differential equation. Therefore $pq = 0$. We choose $q = 0$, $p \neq 0$, then equation (29) becomes:

(30) $p_0(z) = \alpha\beta z^2 - \alpha\beta pz$.

Substituting equations (26), (28) and (30) into equation (9) gives us the following equation:

$$(31)\quad (z^4 - z^3)\frac{d^2u}{dz^2} + [(1+\alpha+\beta)z^3 - (1+\alpha+\beta-\delta)z^2]\frac{du}{dz} + \alpha\beta z(z-p)u = 0, \text{ or}$$

upon slight simplification, equation (31) reduces to:

$$(32)\quad z^2(z-1)\frac{d^2u}{dz^2} + [(1+\alpha+\beta)z^2 - (1+\alpha+\beta-\delta)z]\frac{du}{dz} + \alpha\beta(z-p)u = 0.$$

Equation (32) was derived on the basis of choosing $a = b = q = 0$, $\gamma p \neq 0$. Other choices will produce other differential equations.

We will now classify equation (32) in the notation of Ince [5, p. 497].

Equation (32) can be written as follows:

$$(33)\qquad \frac{d^2u}{dz^2} + [(1+\alpha+\beta)/(z-1) - (1+\alpha+\beta-\delta)/(z^2-z)]\frac{du}{dz} + [\alpha\beta(z-p)/(z^3-z^2)]u = 0, \text{ or}$$

$$(34)\qquad \frac{d^2u}{dz^2} + p(z)\frac{du}{dz} + q(z)u = 0, \text{ where}$$

$$(35)\qquad p(z) = [(1+\alpha+\beta)/(z-1) - (1+\alpha+\beta-\delta)/(z^2-z)], \text{ and}$$

$$(36)\qquad q(z) = [\alpha\beta(z-p)/(z^3-z^2)].$$

We note from equations (35) and (36) that

$$(37)\qquad zp(z) = [z(1+\alpha+\beta)/(z-1) - (1+\alpha+\beta-\delta)/(z-1)], \text{ and}$$

$$(38)\qquad z^2q(z) = [\alpha\beta(z-p)/(z-1)].$$

By saying an equation is analytic at a point, we shall mean the function defined by the equation is analytic at that point. Subject to this definition, we see that equations (37) and (38) are both analytic at $z = 0$, hence we conclude that $z = 0$ is a regular singular point of equation (33).

Again from equations (35) and (36) we have:

$$(39)\qquad (z-1)p(z) = [(1+\alpha+\beta) - (1+\alpha+\beta-\delta)/z], \text{ and}$$

$$(40)\qquad (z-1)^2q(z) = (z-1)(\alpha\beta)(z-p)/z^2.$$

Now since equations (39) and (40) are both analytic at $z = 1$, we conclude that $z = 1$ is also a regular singular point of equation (33).

In order to investigate the point at infinity, we first make the variable transformation:

$$(41)\qquad v = 1/z.$$

We then consider the transformed differential equation:

$$(42) \qquad \frac{d^2u}{dv^2} + [2/v - (1/v^2)p(1/v)]\frac{du}{dv} + (1/v^4)q(1/v)u = 0.$$

In Chapter I we noted that if

$v[2/v - (1/v^2)p(1/v)]$ and $v^2[(1/v^4)q(1/v)]$ are both analytic at $v = 0$, then $v = 0$ is a regular singular point of equation (42). This implies that infinity is a regular singular point of equation (33). Accordingly, from equation (42) we have:

$$(43) \qquad v[2/v - (1/v^2)p(1/v)] = 2 - [(1+\alpha+\beta)/(1-v) - v(1+\alpha+\beta-\delta)/(1-v)],$$

and

$$(44) \qquad v^2[(1/v^4)q(1/v)] = \frac{\alpha\beta(1-pv)}{1-v}.$$

Since equations (43) and (44) are analytic at $v = 0$, we conclude that $v = 0$ is a regular singular point of equation (42), and hence infinity is a regular singular point of equation (33).

We have now tested all points at which equation (33) fails to be analytic, and thus conclude that equation (33) has no irregular singular points. We will now determine whether or not the point $z = 0$, 1, and infinity are elementary singular points [5, p. 497].

Let a solution of equation (32) in a neighborhood of $z = 0$ be written in the form:

$$(45) \qquad u(z) = c_0 z^{\lambda} + c_1 z^{\lambda+1} + \cdots + c_r z^{\lambda+r} + \cdots ; \; c_0 \neq 0.$$

Differentiate equation (45) with respect to z to get:

$$(46) \qquad u'(z) = c_0 \lambda z^{\lambda-1} + c_1(\lambda+1) z^{\lambda} + \cdots + c_r(\lambda+r) z^{\lambda+r-1} + \cdots .$$

Differentiate equation (46) with respect to z to get:

$$(47) \qquad u''(z) = c_0 \lambda(\lambda-1) z^{\lambda-2} + c_1(\lambda+1)(\lambda) z^{\lambda-1} + \cdots$$

$$+ c_r(\lambda+r)(\lambda+r-1) z^{\lambda+r-2} + \cdots .$$

Substitute equations (45) - (47) into equation (32), and equate the coefficient of $z^{\lambda}$ to zero to get the following indicial equation:

$$(48) \qquad -c_0 \lambda(\lambda-1) - c_0 \lambda(1+\alpha+\beta-\delta) - c_0 \alpha\beta p = 0.$$

The values of $\lambda$ which satisfy equation (48) are the exponents of equation (45) at $z = 0$.

Solving equation (48) for $\lambda$ we get:

$$\lambda_1 = \frac{\delta - \alpha - \beta - R}{2} , \text{ and}$$

$$\lambda_2 = \frac{\delta - \alpha - \beta + R}{2} , \text{ where } R = \sqrt{(\alpha+\beta-\delta)^2 - 4\alpha\beta p}.$$

From the above expressions we note that

$$\lambda_2 - \lambda_1 = \sqrt{(\alpha+\beta-\delta)^2 - 4\alpha\beta p} , \text{ and hence } \lambda_2 - \lambda_1 \neq \frac{1}{2} .$$

We thus conclude that $z = 0$ is not an elementary singularity of equation (33).

To investigate whether or not $z = 1$ is an elementary singularity, we first make the variable transformation $v = z - 1$, whence $z = v + 1$. Equation (32) now becomes:

$$(49)\quad v(v^2+2v+1)\frac{d^2u}{dv^2} + [(1+\alpha+\beta)(v^2+2v+1) - (1+\alpha+\beta-\delta)(v+1)]\frac{du}{dv} + \alpha\beta(v+1-p)u = 0.$$

Proceeding as before, we obtain the following indicial equation when we equate the coefficient of $v^{\lambda-1}$ to zero:

$$(50)\quad c_0\lambda(\lambda-1) + c_0\lambda[(1+\alpha+\beta) - (1+\alpha+\beta-\delta)] = 0.$$

Solving equation (50) we get

$$\lambda_1 = 0, \text{ and}$$

$$\lambda_2 = 1 - \delta.$$

Since $\lambda_2 - \lambda_1 \neq 1/2$, we conclude that $z = 1$ is not an elementary singularity.

We finally investigate whether or not infinity is an elementary singularity. First we let

$$(51)\quad v = 1/z. \text{ Then}$$

$$(52)\quad \frac{du}{dz} = - v^2 \frac{du}{dv}, \text{ and}$$

$$(53)\quad \frac{d^2u}{dz^2} = v^4 \frac{d^2u}{dv^2} + 2v^3 \frac{du}{dv}.$$

Substituting equations (51) - (53) into equation (32) gives the following equation:

$$(54)\qquad \frac{1}{v^2}\left(\frac{1}{v}-1\right)\left[v^4\frac{d^2u}{dv^2}+2v^3\frac{du}{dv}\right]$$
$$+\left[(1+\alpha+\beta)(1/v^2)-(1+\alpha+\beta-\delta)(1/v)\right]\left[-v^2\frac{du}{dv}\right]$$
$$+\alpha\beta\left(\frac{1}{v}-p\right)u=0.$$

Simplification of equation (54) yields:

$$(55)\qquad v^2(1-v)\frac{d^2u}{dv^2}+\left[2v(1-v)-v(1+\alpha+\beta)+v^2(1+\alpha+\beta-\delta)\right]\frac{du}{dv}$$
$$+\alpha\beta(1-vp)u=0.$$

Proceeding as before, we obtain the following indicial equation upon equating the coefficient of $v^\lambda$ to zero:

$$(56)\qquad c_o\lambda(\lambda-1)+c_o\lambda[2-1-\alpha-\beta]+c_o\alpha\beta=0;\ c_o\neq 0.$$

Solving equation (56) gives:

$$\lambda_1=\alpha, \text{ and}$$
$$\lambda_2=\beta.$$

Consequently $\lambda_2-\lambda_1\neq 1/2$, so that infinity is not an elementary singularity.

To summarize, equation (32) has no elementary singularities, three regular singular points, and no irregular singular points. We therefore classify equation (32) in the notation of Ince as an equation of the [0,3,0] class.

We will now investigate Case II.

Case II: $h = 2$; $m = 0$

For these choices of h and m, identity (23) becomes:

(57) $p_2(z) = S_2z^2 - T_2z^4 \equiv z^4 - z^3(a+b+1) + z^2(a+b+ab) - abz.$

Equating coefficients of like powers of z, we get the following set of equations:

(i) $T_2 = -1$,

(ii) $S_2 = a+b+ab$,

(iii) $a+b+1 = 0$, and

(iv) $ab = 0$.

From equation (iv) we can choose $a = 0$, $b \neq 0$; $b = 0$, $a \neq 0$; or $a = b = 0$. The latter choice produces an inconsistency in equations (ii) and (iii), hence we choose $a = 0$, $b \neq 0$. From equation (iii), when $a = 0$, $b = -1$, and thus from equation (ii), $S_2 = -1$. Equation (57) then becomes:

(58) $p_2(z) = z^4 - z^2$.

Comparing equations (11) and (13) when $j = 1$, $h = 2$, and $m = 0$, we deduce the following expression:

(59) $p_1(z) = S_1z - T_1z^3 \equiv A_1z^3 + A_2z^2 + A_3z + A_4$.

Equating coefficients of like powers of z in identity (59) we obtain the following set of equations:

(i) $T_1 = -A_1$,

(ii) $S_1 = A_3$,

(iii) $A_2 = 0$, and

(iv) $A_4 = 0$, where $A_i$, $(i = 1,2,3,4)$, are given by equations (2) - (5).

Now for a = 0 and b = -1, equations (i) and (ii) become:

$T_1 = -(1+\alpha+\beta)$,

$S_1 = -\gamma - \epsilon$, and substitution into equation (59) gives the following expression:

(60) $p_1(z) = (1+\alpha+\beta)z^3 - (\gamma+\epsilon)z$.

Finally, for j = 0, h = 2, and m = 0 we have, upon comparing equations (12) and (13), the following identity:

(61) $p_o(z) = S_o - T_o z^2 \equiv B_1 z^2 + B_2 z + B_3$, from which we get the following equations upon equating coefficients of like powers of z:

(i) $T_o = -B_1$,

(ii) $S_o = B_3$, and

(iii) $B_2 = 0$.

Now $\alpha\beta \neq 0$, hence in equation (ii) p = -q, and equations (i) and (ii) become:

$T_o = -\alpha\beta$,

$S_o = -\alpha\beta p^2$, and substitution into equation (61) gives the following equation:

(62) $p_o(z) = \alpha\beta z^2 - \alpha\beta p^2$.

Substituting equations (58), (60), and (62) into equation (9) gives the following equation:

$$(63) \qquad z^2(z^2-1)\frac{d^2u}{dz^2} + [(1+\alpha+\beta)z^3 - (\gamma+\epsilon)z]\frac{du}{dz} + \alpha\beta(z^2-p^2)u = 0.$$

Equation (63) was derived on the basis of choosing $a = 0$, $b = -1$, and $p = -q$. Other choices, such as $a = -1$, $b = 0$, and $q = -p$, would yield other differential equations. Since equation (63) is a confluent form, we will not study it any further at this time. However, we note in passing that equation (63) can be classified in a manner analagous to that which was used to classify equation (32).

We will now investigate Case III.

Case III: $h = 3$; $m = 1$

For these choices of h and m, identity (23) becomes:

(64) $p_2(z) = S_2z - T_2z^4 \equiv z^4 - z^3(a+b+1) + z^2(a+b+ab) - abz.$

Again, equating coefficients of like powers of z gives the following set of equations:

(i) $T_2 = -1$,

(ii) $S_2 = -ab$,

(iii) $a+b+1 = 0$, and

(iv) $a+b+ab = 0$.

Solving equations (iii) and (iv) simultaneously gives the following numerical values for a and b:

$a_1 = \omega$; $b_1 = \omega^2$; and

$a_2 = \omega^2$; $b_2 = \omega$, where 1, $\omega$, and $\omega^2$ are the cube roots of unity, such that

(65) $1 + \omega + \omega^2 = 0.$

We choose $a_1 = \omega$ and $b_1 = \omega^2$, then equation (ii) becomes:

$S_2 = -1$, also $T_2 = -1$. Substituting these values into equation (64) gives us:

(66) $p_2(z) = z^4 - z.$

We now compare equations (11) and (13) where $j = 1$, $h = 3$, and $m = 1$, to obtain:

(67) $p_1(z) = S_1 - T_1z^3 \equiv A_1z^3 + A_2z^2 + A_3z + A_4.$

Equating coefficients of like powers of z in identity (67) yields:

(i) $T_1 = -A_1$,

(ii) $S_1 = A_4$,

(iii) $A_2 = 0$, and

(iv) $A_3 = 0$.

Now from the above choices for a and b, $A_4 = -\gamma$, hence we have immediately from equation (67)

(68) $p_1(z) = (1+\alpha+\beta)z^3 - \gamma$.

Again, we compare equations (12) and (13) where $j = 0$, $h = 3$, and $m = 1$ to get:

(69) $p_0(z) = S_0z^{-1} - T_0z^2 \equiv B_1z^2 + B_2z + B_3$.

Proceeding as above, we equate coefficients of like powers of z in identity (69) to obtain:

(i) $T_0 = -B_1$,

(ii) $S_0 = 0$,

(iii) $B_2 = 0$, and

(iv) $B_3 = 0$.

Since $B_2 = 0$, we conclude that $p + q = 0$ since $\alpha\beta \neq 0$, hence $p = -q$. Further since $B_3 = \alpha\beta pq = 0$, then $p = -q = 0$.

Equation (69) then becomes:

(70) $p_0(z) = \alpha\beta z^2$.

Substituting equations (66), (68), and (70) into equation (9) gives the desired non-confluent differential equation:

$$(71)\qquad (z^4-z)\frac{d^2u}{dz^2} + [(1+\alpha+\beta)z^3 - \gamma]\frac{du}{dz} + \alpha\beta z^2 = 0.$$

Equation (71) was derived by letting $a = \omega$, $b = \omega^2$, and $p = q = 0$, where $\omega = \frac{-1 + i\sqrt{3}}{2}$, and $i^2 = -1$. We classify equation (71) as an equation of the [0,5,0] class, since it contains no elementary singularities, and no irregular singular points.

We will now solve equation (71), by first assuming a solution of the form given by equation (18), with $w \equiv z$. We next substitute equation (66) into equation (20), substitute equation (68) into equation (19), and equation (70) into equation (18). The result follows:

$$\begin{aligned}(72)\qquad [z^4-z][&c_0\lambda(\lambda-1)z^{\lambda-2} + c_3(\lambda+3)(\lambda+2)z^{\lambda+1} + \cdots \\ &+ c_{3r}(\lambda+3r)(\lambda+3r-1)z^{\lambda+3r-2} \\ &+ c_{3r+3}(\lambda+3r+3)(\lambda+3r+2)z^{\lambda+3r+1} + \cdots] \\ &+ [(1+\alpha+\beta)z^3-\gamma][c_0\lambda z^{\lambda-1} + c_3(\lambda+3)z^{\lambda+2} + \cdots \\ &+ c_{3r}(\lambda+3r)z^{\lambda+3r-1} + c_{3r+3}(\lambda+3r+3)z^{\lambda+3r+2} + \cdots] \\ &+ [\alpha\beta z^2][c_0z^{\lambda} + c_3z^{\lambda+3} + \cdots + c_{3r}z^{\lambda+3r} \\ &+ c_{3r+3}z^{\lambda+3r+3} + \cdots] = 0.\end{aligned}$$

We obtain the indicial equation by equating the coefficient of $z^{\lambda-1}$ to zero:

(73) $-c_0\lambda(\lambda-1) - \gamma(c_0\lambda) = 0$, so that for $c_0 \neq 0$, $\lambda = 0, 1-\gamma$. This is a verification that the exponents at $z = 0$ are $\lambda = 0, 1-\gamma$, as chosen.

To obtain the recurrence formula, we equate the coefficient of $z^{\lambda+3r+2}$ to zero:

(74) $$c_{3r}(\lambda+3r)(\lambda+3r-1) - c_{3r+3}(\lambda+3r+3)(\lambda+3r+2) + (1+\alpha+\beta)(c_{3r})(\lambda+3r) - \gamma c_{3r+3}(\lambda+3r+3) + \alpha\beta c_{3r} = 0.$$

Solving equation (74) for $c_{3r+3}$ yields:

(75) $$c_{3r+3} = \frac{[(\lambda+3r)(\lambda+3r-1) + (1+\alpha+\beta)(\lambda+3r) + \alpha\beta]}{(\lambda+3r+3)(\lambda+3r+2) + \gamma(\lambda+3r+3)} c_{3r}.$$

Upon slight simplification, equation (75) becomes:

(76) $$c_{3r+3} = \frac{(\lambda+3r+\alpha)(\lambda+3r+\beta)}{(\lambda+3r+3)(\lambda+3r+2+\gamma)} c_{3r}.$$

In equation (76), let $r = 0$, then

(77) $c_3 = \frac{(\lambda+\alpha)(\lambda+\beta)}{(\lambda+3)(\lambda+2+\gamma)} c_0$. Next, let $r = 1$ in equation (76), hence

(78) $$c_6 = \frac{(\lambda+3+\alpha)(\lambda+3+\beta)}{(\lambda+3+3)(\lambda+3+2+\gamma)} c_1.$$

We now substitute equation (77) into equation (78) to get:

(79) $$c_6 = \frac{(\lambda+\alpha)(\lambda+\alpha+3)(\lambda+\beta)(\lambda+\beta+3)}{(\lambda+3)(\lambda+3+3)(\lambda+2+\gamma)(\lambda+2+\gamma+3)} c_0.$$

We now generalize by letting r = n-1 in equation (76) to get:

$$(80)\quad c_{3n} = \frac{(\lambda+\alpha)(\lambda+\alpha+3)\cdots(\lambda+\alpha+3n-3)(\lambda+\beta)(\lambda+\beta+3)\cdots(\lambda+\beta+3n-3)}{(\lambda+3)(\lambda+3+3)\cdots(\lambda+3+3n-3)(\lambda+2+\gamma)\cdots(\lambda+2+\gamma+3n-3)}\, c_o.$$

Dividing numerator and denominator of equation (80) by $3^{2n}$, we obtain:

$$(81)\quad c_{3n} = \frac{(\frac{\lambda+\alpha}{3})(\frac{\lambda+\alpha}{3}+1)\cdots(\frac{\lambda+\alpha}{3}+n-1)(\frac{\lambda+\beta}{3})(\frac{\lambda+\beta}{3}+1)\cdots(\frac{\lambda+\beta}{3}+n-1)}{(\frac{\lambda+3}{3})(\frac{\lambda+3}{3}+1)\cdots(\frac{\lambda+3}{3}+n-1)(\frac{\lambda+2+\gamma}{3})\cdots(\frac{\lambda+2+\gamma}{3}+n-1)}\, c_o.$$

We now set $\lambda = 0$ in equation (81), and substitute the result into equation (17), where equation (17) is written as follows:

$(82)\quad \bar{u}(w) = c_o w^{\lambda}[1 + \sum_{n=1}^{\infty} \frac{c_{3n}}{c_o} w^{3n}]$. We then get:

$$(83)\quad u(w) = K_1[1 + \sum_{n=1}^{\infty} \frac{(\frac{\alpha}{3})(\frac{\alpha}{3}+1)\cdots(\frac{\alpha}{3}+n-1)(\frac{\beta}{3})(\frac{\beta}{3}+1)\cdots(\frac{\beta}{3}+n-1)}{n!(\frac{\gamma+2}{3})(\frac{\gamma+2}{3}+1)\cdots(\frac{\gamma+2}{3}+n-1)} w^{3n}],$$

provided $\gamma \neq 1 - 3n$, $(n = 1,2,\ldots)$.

We now recall that the expression for the hypergeometric function is:

$$(84)\quad F(a,b;c;z) = 1 + \sum_{n=1}^{\infty} \frac{(a)(a+1)\cdots(a+n-1)(b)(b+1)\cdots(b+n-1)}{n!(c)(c+1)\cdots(c+n-1)} z^n.$$

Comparing equations (83) and (84), we can express $u(w)$ in terms of the hypergeometric function. We first establish the following correspondences:

$a = \alpha/3, \quad c = (\gamma+2)/3,$

$b = \beta/3$, and $z = w^3$.

Equation (83) now becomes:

(85) $u_1(z) = K_1 F(\frac{\alpha}{3}, \frac{\beta}{3}; \frac{\gamma+2}{3}; z^3); \gamma \neq 1-3n, (n = 1,2,\ldots).$

Similarly, we let $\lambda = 1 - \gamma$ in equation (81) to get:

$$(86) \quad c_{3n} = \frac{(\frac{1-\gamma+\alpha}{3})\cdots(\frac{1-\gamma+\alpha}{3}+n-1)(\frac{1-\gamma+\beta}{3})\cdots(\frac{1-\gamma+\beta}{3}+n-1)}{n!(\frac{1-\gamma+3}{3})(\frac{1-\gamma+3}{3}+1)\cdots(\frac{1-\gamma+3}{3}+n-1)} c_o.$$

Substituting this expression into equation (82), and letting $z \equiv w$ produces the following expression:

(87) $u_2(z) = (K_2 z^{1-\gamma})(S)$, where

$$S = 1 + \sum_{n=1}^{\infty} \frac{(\frac{1-\gamma+\alpha}{3})\cdots(\frac{1-\gamma+\alpha}{3}+n-1)(\frac{1-\gamma+\beta}{3})\cdots(\frac{1-\gamma+\beta}{3}+n-1)}{n!(\frac{1-\gamma+3}{3})(\frac{1-\gamma+3}{3}+1)\cdots(\frac{1-\gamma+3}{3}+n-1)} z^{3n}.$$

Again comparing equations (84) and (87), we can express $u_2(z)$ in terms of the hypergeometric function, thus:

(88) $u_2(z) = K_2 z^{1-\gamma} F(\frac{1-\gamma+\alpha}{3}, \frac{1-\gamma+\beta}{3}; \frac{4-\gamma}{3}; z^3)$, provided

$\gamma \neq 3n + 1, (n = 1,2,\ldots).$

A general solution of equation (71) in a neighborhood of z = 0 is a linear combination of equations (85) and (88). That is,

$$(89) \qquad u(z) = K_3 F[\tfrac{\alpha}{3}, \tfrac{\beta}{3}; \tfrac{\gamma+2}{3}; z^3]$$

$$+ K_4 z^{1-\gamma} F[\tfrac{1-\gamma+\alpha}{3}, \tfrac{1-\gamma+\beta}{3}; \tfrac{4-\gamma}{3}; z^3],$$

provided $\gamma \neq 1 \pm 3n$, $(n = 1,2,\ldots)$.

From the ratio test, it can be shown that the expression for u(z), as given by the right member of equation (89), is absolutely convergent provided $|z| < 1$.

This completes our analysis of the application of Scheffé's criteria to equation (1) when we investigate a solution in a neighborhood of z = 0. We will next apply Scheffé's criteria [13] to equation (1) and investigate a solution of the result in a neighborhood of z = 1.

Solution in a Neighborhood of z = 1

In order to apply Scheffé's criteria to equation (1) and consider a solution of the resulting equation in a neighborhood of z = 1, we first make the variable transformation:

(90) $w = z - 1$, or

(91) $z = w + 1$.

From equations (90) and (91) it follows that

(92) $\frac{du}{dw} = \frac{du}{dz}$, and

(93) $\frac{d^2u}{dw^2} = \frac{d^2u}{dz^2}$.

Substituting equations (91) - (93) in equation (1) gives the following equation:

(94) $$(w+1)(w)(w+1-a)(w+1-b)\frac{d^2u}{dw^2} + [A_1(w+1)^3 + A_2(w+1)^2 + A_3(w+1) + A_4]\frac{du}{dw} + [B_1(w+1)^2 + B_2(w+1) + B_3]u = 0.$$

We can write equation (94) as follows:

(95) $p_2(w)\frac{d^2u}{dw^2} + p_1(w)\frac{du}{dw} + p_0(w)u = 0$, where

(96) $p_2(w) = (w+1)(w)(w+1-a)(w+1-b)$,

(97) $p_1(w) = [A_1(w+1)^3 + A_2(w+1)^2 + A_3(w+1) + A_4]$, and

(98) $p_0(w) = [B_1(w+1)^2 + B_2(w+1) + B_3]$.

We will now express each of the above coefficients as a polynomial in w, thus from equation (96):

$$p_2(w) = (w^2+w)(w^2+w-bw+w+1-b-aw-a+ab)$$
$$= (w^2+w)[w^2+w(2-a-b) + (ab-a-b+1)]$$
$$= w^4 + w^3(2-a-b) + w^2(ab-a-b+1)$$
$$+ w^3 + w^2(2-a-b) + w(ab-a-b+1)$$
$$= w^4 + w^3(2-a-b+1) + w^2(ab-a-b+3-a-b)$$
$$+ w(a-1)(b-1), \text{ or finally,}$$

$$(99) \quad p_2(w) = w^4 + w^3(3-a-b) + w^2[(a-2)(b-2)-1]$$
$$+ w(a-1)(b-1).$$

Similarly from equation (97):

$$p_1(w) = A_1(w^3+3w^2+3w+1) + A_2(w^2+2w+1)$$
$$+ A_3(w+1) + A_4, \text{ or}$$

$$(100) \quad p_1(w) = A_1w^3 + w^2(3A_1+A_2) + w(3A_1+2A_2+A_3)$$
$$+ (A_1+A_2+A_3+A_4).$$

Similarly, equation (98) may be written:

$$(101) \quad p_0(w) = B_1w^2 + w(2B_1+B_2) + (B_1+B_2+B_3).$$

Substituting equations (99) - (101) into equation (95) gives us the differential equation in the desired form, namely:

$$(102) \quad \{w^4 + w^3(3-a-b) + w^2[(a-2)(b-2)-1] + w(a-1)(b-1)\}\frac{d^2u}{dw^2}$$
$$+ [A_1w^3 + w^2(3A_1+A_2) + w(3A_1+2A_2+A_3)$$
$$+ (A_1+A_2+A_3+A_4)]\frac{du}{dw}$$
$$+ [B_1w^2 + w(2B_1+B_2) + (B_1+B_2+B_3)]u = 0.$$

Prior to applying Scheffé's criteria, we substitute equation (91) into equation (13), let $z_0 = 1$, and replace $p_j$ by $\bar{p}_j$ to get:

(103) $\bar{p}_j(w+1) = [S_j - T_j w^h]w^{j-m}$.

Letting $p_j(w) = \bar{p}_j(w+1)$, equation (103) becomes:

(104) $p_j(w) = [S_j - T_j w^h]w^{j-m}$, $(j = 0,1,2)$.

We initially let $j = 2$, then from equations (99) and (104) we obtain the following identity:

(105) $$p_2(w) = [S_2 - T_2 w^h]w^{2-m}$$
$$\equiv w^4 + w^3(3-a-b) + w^2[(a-2)(b-2)-1] + w(a-1)(b-1).$$

We assume $T_2 = 0$, so that equation (105) becomes:

(106) $$S_2 w^{2-m} \equiv w^4 + w^3(3-a-b) + w^2[(a-2)(b-2)-1] + w(a-1)(b-1).$$

From identity (106) we surmise $2 - m = 4$; otherwise, the coefficient of $w^4$ in the left member of identity (106) is zero while the coefficient of $w^4$ in the right member is unity. This is an obvious inconsistency. Since $2 - m = 4$, we get the following set of equations:

(i) $S_2 = 1$,

(ii) $3-a-b = 0$,

(iii) $(a-2)(b-2)-1 = 0$, and

(iv) $(a-1)(b-1) = 0$.

In equation (iv) above, let a = 1. Then equation (iii) implies b = 1. But from equation (ii), a = b = 1 implies 1 = 0. This inconsistency leads us to the conclusion $T_2 \neq 0$.

Again in equation (105), we assume $S_2 = 0$. We then get the following identity:

$$(107) \quad -T_2 w^{h+2-m} \equiv w^4 + w^3(3-a-b) + w^2[(a-2)(b-2)-1] + w(a-1)(b-1).$$

Reasoning as before, we note that $h+2-m = 4$. We then get the following set of equations:

(i) $T_2 = -1$,

(ii) $3-a-b = 0$,

(iii) $(a-2)(b-2)-1 = 0$, and

(iv) $(a-1)(b-1) = 0$.

Since equations (ii) - (iv) in the above set are the same as equations (ii) - (iv) in the preceding set, which were found to be inconsistent, we conclude that $S_2 \neq 0$. Thus $S_2 T_2 \neq 0$.

From identity (105) we have the following:

$$(108) \quad S_2 w^{2-m} - T_2 w^{h+2-m} \equiv w^4 + w^3(3-a-b) + w^2[(a-2)(b-2)-1] + w(a-1)(b-1).$$

From this identity we notice that $h+2-m = 4 > 2-m$, since h is a positive integer. We can now deduce a set of values of h and m which will be shown to be exhaustive.

Since $h+2-m = 4$, then $h - m = 2$. We then have the following cases to consider:

Case I: $h = 1$; $m = -1$,

Case II: $h = 2$; $m = 0$, and

Case III: $h = 3$; $m = 1$.

We reiterate that h is a positive integer, consequently $m \geq -1$. If $h \geq 4$, then $m \geq 2$, and from identity (108), $S_2 = 0$. This has been demonstrated to be impossible. We therefore conclude that the above set is exhaustive. We now investigate each case in turn.

Case I: h = 1; m = -1

For h = 1 and m = -1, identity (108) becomes:

$$(109)\quad S_2w^3 - T_2w^4 \equiv w^4 + w^3(3-a-b) + w^2[(a-2)(b-2)-1] + w(a-1)(b-1).$$

From identity (109) we get the following set of equations:

(i) $T_2 = -1$,

(ii) $S_2 = 3-a-b$,

(iii) $(a-2)(b-2)-1 = 0$, and

(iv) $(a-1)(b-1) = 0$.

From equations (iii) and (iv), we note that a = b = 1. Hence $S_2 = 1$, and we have for $p_2(w)$:

$$(110)\quad p_2(w) = w^4 + w^3.$$

In equation (104), we set j = 1, h = 1, and m = -1, and compare the result with equation (100):

$$(111)\quad p_1(w) = [S_1w^2 - T_1w^3] \equiv A_1w^3 + w^2(3A_1+A_2) + w(3A_1+2A_2+A_3) + (A_1+A_2+A_3+A_4).$$

From identity (111) we obtain the following set of equations upon equating coefficients of like powers of w:

(i) $T_1 = -A_1$,

(ii) $S_1 = 3A_1 + A_2$,

(iii) $3A_1+2A_2+A_3 = 0$, and

(iv) $A_1+A_2+A_3+A_4 = 0$.

From equations (ii) - (iv), we deduce that $3A_1 + A_2 = A_1 + A_4$. By equations (2) and (5), this reduces to $(1+\alpha+\beta) - ab\gamma$. Then $T_1 = -(1+\alpha+\beta)$, $S_1 = 1+\alpha+\beta-ab\gamma$, and equation (111) becomes, for $a = b = 1$:

(112) $p_1(w) = (1+\alpha+\beta)w^3 + (1+\alpha+\beta-\gamma)w^2$.

To compute $p_0(w)$, we compare equations (101) and (104) with $j = 0$, $h = 1$, and $m = -1$. The result follows:

(113) $p_0(w) = S_0w - T_0w^2 \equiv B_1w^2 + w(2B_1 + B_2) + (B_1+B_2+B_3)$.

Equating coefficients of like powers of w in identity (113) yields the following set of equations:

(i) $T_0 = -B_1$,

(ii) $S_0 = 2B_1 + B_2$, and

(iii) $B_1+B_2+B_3 = 0$.

From equations (6) - (8), $B_1 + B_2 + B_3 = \alpha\beta(1-p)1-q) = 0$. But $\alpha\beta \neq 0$, or else the differential equation is essentially first order. We therefore choose $p = 1$. For this choice of p, $S_0 = 2\alpha\beta - \alpha\beta(q+1)$, or $S_0 = \alpha\beta(1-q)$. Also from equation (6), $B_1 = \alpha\beta$, so that $T_0 = -\alpha\beta$. Substituting these values of $S_0$ and $T_0$ into equation (113) gives us the following expression for $p_0(w)$:

(114) $p_0(w) = \alpha\beta w^2 + \alpha\beta(1-q)w$.

Substituting equations (110), (112), and (114) into equation (95) gives the following equation:

$$(115)\quad (w^4+w^3)\frac{d^2u}{dw^2} + [(1+\alpha+\beta)w^3 + (1+\alpha+\beta-\gamma)w^2]\frac{du}{dw} + [\alpha\beta w^2 + \alpha\beta(1-q)w]u = 0.$$

Equation (115) reduces to the following after simplification:

$$(116)\quad w^2(w+1)\frac{d^2u}{dw^2} + [(1+\alpha+\beta)w^2 + (1+\alpha+\beta-\gamma)w]\frac{du}{dw} + \alpha\beta[w + (1-q)]u = 0.$$

Equation (116) was derived by choosing a = b = p = 1. Choices for a and b are fixed; however, we can choose q = 1, in which case the resulting equation looks like equation (116) with q replaced by p. We will now classify equation (116) in the notation of Ince.

We may write equation (116) as follows:

$$(117)\quad \frac{d^2u}{dw^2} + p(w)\frac{du}{dw} + q(w)u = 0, \text{ where}$$

$$(118)\quad p(w) = \frac{1+\alpha+\beta}{w+1} + \frac{(1+\alpha+\beta-\gamma)}{w(w+1)}, \text{ and}$$

$$(119)\quad q(w) = \frac{\alpha\beta(w+1-q)}{w^2(w+1)}.$$

From equations (118) and (119) we note that

$$(120)\quad wp(w) = \frac{w(1+\alpha+\beta)}{w+1} + \frac{(1+\alpha+\beta-\gamma)}{w+1}, \text{ and}$$

$$(121)\quad w^2q(w) = \frac{\alpha\beta(w+1-q)}{w+1}.$$

Equations (120) and (121) are analytic at w = 0, that is at z = 1, hence w = 0 is a regular singular point of equation (116). Similarly we note

(122) $(w+1)p(w) = (1+\alpha+\beta) + \frac{(1+\alpha+\beta-\gamma)}{w}$, and

(123) $(w+1)^2 q(w) = \frac{\alpha\beta(w+1-q)(w+1)}{w^2}$ are both analytic at w = -1, that is at z = 0, hence w = -1 is a regular singular point of equation (116).

In order to investigate the point at infinity, we first make the variable transformation:

(124) $v = \frac{1}{w}$. Equation (117) then becomes:

(125) $\frac{d^2u}{dv^2} + [2/v - \frac{1}{v^2} p(1/v)]\frac{du}{dv} + \frac{1}{v^4} q(1/v)u = 0.$

From equations (118) and (124), we conclude that

(126) $p(1/v) = \frac{v(1+\alpha+\beta)}{v+1} + \frac{v^2(1+\alpha+\beta-\gamma)}{v+1}.$

Similarly, upon comparing equations (119) and (124), we note that

(127) $q(1/v) = \frac{v^2\alpha\beta(1+v-qv)}{v+1}$ .

From equations (126) and (127) we easily deduce the following equations:

(128) $[2/v - \frac{1}{v^2} p(1/v)] = 2/v - \frac{(1+\alpha+\beta)}{v(v+1)} - \frac{(1+\alpha+\beta-\gamma)}{v+1}$, and

(129) $\frac{1}{v^4} q(1/v) = \frac{\alpha\beta(1+v-qv)}{v^2(v+1)}.$

The theory developed in Chapter I shows that if both $v[2/v - \frac{1}{v^2}\, p(1/v)]$ and $(v^2/v^4)q(1/v)$ are analytic at $v = 0$, then $v = 0$ is a regular singular point of equation (125). This implies, by way of equation (124), that infinity is a regular singular point of equation (116). From equations (128) and (129), we get:

$$(130) \qquad v[2/v - \frac{1}{v^2}\, p(1/v)] = 2 - \frac{1+\alpha+\beta}{v+1} - \frac{(1+\alpha+\beta-\gamma)v}{v+1}, \text{ and}$$

$$(131) \qquad (v^2/v^4)q(1/v) = \frac{\alpha\beta(1+v-qv)}{v+1}.$$

Equations (130) and (131) are analytic at $v = 0$, which implies that infinity is a regular singular point of equation (116). We will now investigate whether or not $w = 0$, 1, and infinity are elementary singularities.

Let a formal solution of equation (116) be as follows:

$$(132) \qquad u(w) = c_0 w^{\lambda} + c_1 w^{\lambda+1} + \cdots + c_r w^{\lambda+r} + \cdots \; ; \; c_0 \neq 0.$$

Differentiate equation (132) with respect to w to get:

$$(133) \qquad u'(w) = c_0 \lambda w^{\lambda-1} + c_1(\lambda+1)w^{\lambda} + \cdots + c_r(\lambda+r)w^{\lambda+r-1} + \cdots .$$

Differentiate equation (133) with respect to w to get:

$$(134) \qquad u''(w) = c_0 \lambda(\lambda-1)w^{\lambda-2} + c_1(\lambda+1)(\lambda)w^{\lambda-1} + \cdots + c_r(\lambda+r)(\lambda+r-1)w^{\lambda+r-2} + \cdots .$$

Substituting equations (132) - (134) into equation (116), and equating the coefficient of $w^{\lambda}$ to zero, yields the following indicial equation:

(135) $c_o\lambda(\lambda-1) + c_o\lambda(1+\alpha+\beta-\gamma) + c_o(\alpha\beta)(1-q) = 0.$

After simplification, equation (135) reduces to:

(136) $\lambda^2 + \lambda(\alpha+\beta-\gamma) + \alpha\beta(1-q) = 0.$

Let the roots of equation (136) be

$$\lambda_1 = \frac{\gamma-\alpha-\beta-R}{2}, \text{ and}$$

$$\lambda_2 = \frac{\gamma-\alpha-\beta+R}{2}, \text{ where}$$

$$R = \sqrt{(\alpha+\beta-\gamma)^2 - 4\alpha\beta(1-q)}.$$

From the above equations, it immediately follows that

$$\lambda_2 - \lambda_1 = \sqrt{(\alpha+\beta-\gamma)^2 - 4\alpha\beta(1-q)} \neq \frac{1}{2},$$

and thus $w = 0$ is not an elementary singular point. This implies, by way of equation (90), that $z = 1$ is not an elementary singular point.

To consider the point $z = 0$, we first apply equation (90) to equation (116) thus obtaining:

$$(137)\quad z(z-1)^2\frac{d^2u}{dz^2} + [(1+\alpha+\beta)(z-1)^2 + (1+\alpha+\beta-\gamma)(z-1)]\frac{du}{dz} + \alpha\beta(z-q)u = 0.$$

Equation (137) simplifies to the following:

$$(138)\quad z(z^2-2z+1)\frac{d^2u}{dz^2} + [(1+\alpha+\beta)z^2 - (1+\alpha+\beta)z + \gamma]\frac{du}{dz} + \alpha\beta(z-q)u = 0.$$

If we assume a solution of equation (138) to be:

(139) $u(z) = \sum_{r=0}^{\infty} c_r z^{\lambda+r}$; $c_o \neq 0$, then we can immediately deduce the following indicial equation upon equating the coefficient of $z^{\lambda-1}$ to zero:

(140) $c_o\lambda(\lambda-1) + c_o\lambda\gamma = 0$.

From equation (140) it follows that

$\lambda_1 = 0$, and

$\lambda_2 = 1 - \gamma$, thus $\lambda_2 - \lambda_1 = 1 - \gamma \neq 1/2$.

We therefore conclude that $z = 0$ is not an elementary singular point.

In order to investigate the point at infinity, we initially make the following variable transformation:

(141) $z = 1/v$. Equation (137) then becomes equation (125), where $v = 1/z$,

(142) $p(1/v) = (1+\alpha+\beta)v + \frac{(1+\alpha+\beta-\gamma)v^2}{1 - v}$, and

(143) $q(1/v) = \frac{\alpha\beta(1-qv)v^3}{(1-v)^2}$.

We now compute $[2/v - (1/v^2)p(1/v)]$, and $(1/v^4)q(1/v)$, thus obtaining:

(144) $[2/v - (1/v^2)p(1/v)] = \left[\frac{2}{v} - \frac{1+\alpha+\beta}{v} - \frac{1+\alpha+\beta-\gamma}{(1-v)}\right]$.

Similarly,

(145) $(1/v^4)q(1/v) = \frac{\alpha\beta(1-qv)}{v^2(1-v)^2}$.

Substituting equations (144) and (145) into equation (125), and simplifying slightly yields:

$$(146)\quad v^2(1-v)^2\frac{d^2u}{dv^2} + [v(1-v)^2(1-\alpha-\beta) - v^2(1-v)(1+\alpha+\beta-\gamma)]\frac{du}{dv} + \alpha\beta(1-qv)u = 0.$$

Now if $v = 0$ is an elementary singular point of equation (146), then infinity is an elementary singular point of equation (138). We therefore assume a solution of equation (146) to be:

$$(147)\quad u(v) = \sum_{r=0}^{\infty} c_r v^{\lambda+r};\quad c_0 \neq 0.$$

From equations (146) and (147) we can easily deduce the following indicial equation when the coefficient of $v^{\lambda}$ is equated to zero:

$$(148)\quad c_0\lambda(\lambda-1) + c_0\lambda(1-\alpha-\beta) + c_0\alpha\beta = 0.$$

Reducing equation (148) yields the following equation:

$$(149)\quad \lambda^2 - \lambda(\alpha+\beta) + \alpha\beta = 0, \text{ so that}$$

$$\lambda_1 = \alpha, \text{ and}$$

$$\lambda_2 = \beta.$$

From these equations it follows that

$\lambda_2 - \lambda_1 = \beta - \alpha \neq 1/2$, therefore $v = 0$ is not an elementary singular point of equation (146), consequently infinity is not an elementary singular point of equation (138).

We have examined all singular points of equation (138), and have shown it to have no elementary singular points nor irregular singular points, we therefore classify equation (138) as an equation of the [0,3,0] class.

We will now examine Case II.

Case II: h = 2; m = 0

We set j = 2, h = 2, and m = 0 in equation (10⁴), and compare the result with equation (99) to get the following identity:

$$(150)\quad p_2(w) = S_2w^2 - T_2w^4 \equiv w^4 + w^3(3-a-b) + w^2[(a-2)(b-2)-1] + w(a-1)(b-1).$$

Equating coefficients of like powers of w in identity (150) gives the following set of equations:

(i) $T_2 = -1$,

(ii) $S_2 = (a-2)(b-2)-1$,

(iii) $3-a-b = 0$, and

(iv) $(a-1)(b-1) = 0$.

We choose a = 1 in equation (iv). Then from equation (iii), b = 2, and from equation (ii), $S_2 = -1$. The choice b = 1, a = 2, hence $S_2 = -1$, will result in the derivation of another differential equation. Equation (150) now becomes:

$$(151)\quad p_2(w) = w^4 - w^2.$$

Again in equation (10⁴), we let j = 1, h = 2, and m = 0, then compare the result with equation (100) to get the following expression:

$$(152)\quad p_1(w) = S_1w - T_1w^3 \equiv A_1w^3 + w^2(3A_1+A_2) + w(3A_1+2A_2+A_3) + (A_1+A_2+A_3+A_4).$$

Equating coefficients of like powers of w in both members of identity (152) gives us the following equations:

(i) $T_1 = -A_1$,

(ii) $S_1 = 3A_1 + 2A_2 + A_3$,

(iii) $3A_1 + A_2 = 0$, and

(iv) $A_1 + A_2 + A_3 + A_4 = 0$.

From equations (i) - (iv), and from equations (2) - (5), we immediately deduce the following, with a = 1, b = 2:

$T_1 = -(1+\alpha+\beta)$, and

$S_1 = 2\gamma - (1+\alpha+\beta)$.

Substituting these values of $T_1$ and $S_1$ into equation (152) gives us the following equation:

(153) $p_1(w) = (1+\alpha+\beta)w^3 + (2\gamma-1-\alpha-\beta)w$.

We repeat the above process to compute $p_o(w)$. That is, in equation (104), we let j = 0, h = 2, and m = 0, then compare the resulting expression with equation (101) to get:

(154) $p_o(w) = S_o - T_o w^2 \equiv B_1 w^2 + w(2B_1+B_2) + (B_1+B_2+B_3)$.

Proceeding as before, we equate the coefficients of like powers of w in both members of identity (154) to obtain the following set of equations:

(i) $T_o = -B_1$,

(ii) $S_o = B_1+B_2+B_3$, and

(iii) $2B_1 + B_2 = 0$.

From equations (i) - (iii) above, and from equations (6) - (8), we immediately deduce:

$T_o = -\alpha\beta$, and

$S_o = \alpha\beta(1-p)(1-q) = -\alpha\beta(p-1)^2$, since $2B_1 + B_2 = 0$ implies $q = 2 - p$. Substituting these values of $T_o$ and $S_o$ into equation (154) gives us the following equation:

(155) $p_o(w) = \alpha\beta w^2 - \alpha\beta(1-p)^2$.

We now substitute equations (151), (153), and (155) into equation (95) to get the following differential equation:

$$(156)\quad w^2(w^2-1)\frac{d^2u}{dw^2} + [(1+\alpha+\beta)w^3 - (1+\alpha+\beta-2\gamma)w]\frac{du}{dw} + \alpha\beta[w^2 - (1-p)^2]u = 0.$$

Equation (156) was derived by letting $a = 1$, $b = 2$, and $q = 2 - p$. By choosing $a = 2$, $b = 1$, and letting $p = 2 - q$, other differential equations could be similarly derived. Equation (156) may be expressed in terms of z, and derivatives of u with respect to z, by applying the transformation given by equation (90). Further analysis of equation (156) is beyond the scope of this study; however, we do mention that it can be classified in a manner similar to that used to classify equation (116).

We will now study Case III.

Case III: h = 3; m = 1

Letting j = 2, h = 3, and m = 1 in equation (104), then comparing the result with equation (99), gives us the following expression:

(157) $p_2(w) = S_2w - T_2w^4$

$$\equiv w^4 + w^3(3-a-b) + w^2[(a-2)(b-2)-1] + w(a-1)(b-1).$$

Equating coefficients of like powers of w in both members of equation (157), gives us the following set of equations:

(i) $T_2 = -1$,

(ii) $S_2 = (a-1)(b-1)$,

(iii) $3-a-b = 0$, and

(iv) $(a-2)(b-2)-1 = 0$.

We now solve equations (iii) and (iv) simultaneously for a and b. Thus from equation (iii), $b = 3 - a$. From equation (iv), $ab - 2(a+b) + 3 = 0$, hence $ab = 3$, and $a + b = 3$, thus

$a(3-a) = 3$ or $a^2 - 3a + 3 = 0$. So that

$a_1 = 1 - \omega$; $b_1 = 1 - \omega^2$,

$a_2 = 1 - \omega^2$; $b_2 = 1 - \omega$, where $\omega = \frac{-1+i\sqrt{3}}{2}$ and $i^2 = -1$. Another useful relationship from the above equations is $ab - 2(a+b) + (a+b) = 0$, since $a + b = 3$, or $ab - a - b = 0$, so that $a = b/(b-1)$.

We will choose $a_1 = 1-\omega$ and $b_1 = 1-\omega^2$, and insert these values of a and b into the differential equation at the proper time.

We can now compute $S_2$ from equation (ii). Thus $S_2 = ab - (a+b) + 1 = 1$ since $ab = a+b = 3$. Equation (157) now becomes:

(158) $p_2(w) = w^4 + w.$

Again we let $j = 1$, $h = 3$, and $m = 1$ in equation (104), then compare the result with equation (100) to get the following expression:

(159) $p_1(w) = S_1 - T_1 w^3$

$$\equiv A_1 w^3 + w^2(3A_1+A_2) + w(3A_1+2A_2+A_3)$$

$$+ (A_1+A_2+A_3+A_4).$$

As before, we equate the coefficients of like powers of w in both members, to obtain the following equations:

(i) $T_1 = -A_1$,

(ii) $S_1 = A_1+A_2+A_3+A_4$,

(iii) $3A_1+A_2 = 0$, and

(iv) $3A_1+2A_2+A_3 = 0$.

From this set of equations we immediately deduce that $S_1 = A_1 + A_4$, and from equations (2) - (5) we have

$T_1 = -(1+\alpha+\beta)$ and

$S_1 = 1+\alpha+\beta-ab\gamma$. But $ab = 3$, hence

$S_1 = 1+\alpha+\beta-3\gamma$.

Substituting these expressions for $T_1$ and $S_1$ into equation (159), gives us the following expression:

(160) $p_1(w) = (1+\alpha+\beta)w^3 + (1+\alpha+\beta-3\gamma)$.

To compute $p_o(z)$, we proceed in a manner similar to the foregoing, namely let $j = 0$, $h = 3$, and $m = 1$ in equation (104), then compare the result with equation (101), thus:

(161) $p_o(w) = S_o w^{-1} - T_o w^2 \equiv B_1 w^2 + w(2B_1+B_2) + (B_1+B_2+B_3)$.

Equating coefficients of like powers of w gives the following set of equations:

(i) $T_o = -B_1$,

(ii) $S_o = 0$,

(iii) $2B_1+B_2 = 0$, and

(iv) $B_1+B_2+B_3 = 0$.

From equation (iii), and from equations (6) - (8), we have: $2\alpha\beta - \alpha\beta(p+q) = 0$, so that $p + q = 2$. Also from equations (iii) and (iv), we note $B_1 = B_3$, or $\alpha\beta = \alpha\beta pq$. But $\alpha\beta \neq 0$, so that $pq = 1$. Solving $p + q = 2$ and $pq = 1$ simultaneously gives us $p = q = 1$.

Equation (161) now becomes:

(162) $p_o(w) = \alpha\beta w^2$.

Substituting equations (158), (160), and (162) into equation (95) gives us the desired differential equation:

$$(163)\quad w(w^3+1)\frac{d^2u}{dw^2} + [(1+\alpha+\beta)w^3 + (1+\alpha+\beta-3\gamma)]\frac{du}{dw} + \alpha\beta w^2 u = 0.$$

Equation (163) was derived by letting $h = 3$, $m = 1$, $ab = 3$, $a + b = 3$, and $p = q = 1$. From these equations, we choose $a_1 = 1 - \omega$, and $b_1 = 1 - \omega^2$, where $\omega = \frac{-1+i\sqrt{3}}{2}$, and $i^2 = -1$. We also recall that $w = z - 1$. Other choices for a and b will give other differential equations.

Before proceeding with a solution of equation (163), we will first simplify the expression:

$$(164)\quad E = 1+\alpha+\beta-3\gamma.$$

From identity (159) we deduced the following set of equations:

(i) $T_1 = -A_1$,

(ii) $S_1 = A_1+A_2+A_3+A_4$,

(iii) $3A_1+A_2 = 0$, and

(iv) $3A_1+2A_2+A_3 = 0$.

Upon comparing equations (iii) and (iv), we surmise that $A_2 + A_3 = 0$. But from equations (3) and (4), we have:

$$(165)\quad A_2+A_3 = -(1+\alpha+\beta)(1+a+b) + a\epsilon + b\eta + \delta + ab(\gamma+\delta) + a(\gamma+\eta) + b(\gamma+\epsilon) = 0.$$

Simplifying the right member of equation (165), and remembering that ab = a+b = 3, we get:

$$6\gamma + 3\delta + a\eta + b\epsilon = 4(1+\alpha+\beta) - a\epsilon - b\eta - \delta, \text{ or}$$

$$6\gamma + 3\delta + 3\eta + 3\epsilon = 4(1+\alpha+\beta) - \delta.$$

Rearranging the above equation slightly, yields:

$$3(\gamma+\delta+\eta+\epsilon-1-\alpha-\beta) = 1+\alpha+\beta-\delta-3\gamma.$$

However, from equation (12) in Chapter V,

$$\gamma+\delta+\eta+\epsilon-1-\alpha-\beta = 0, \text{ so that}$$

$$1+\alpha+\beta-\delta-3\gamma = 0, \text{ and finally,}$$

(166) $$1+\alpha+\beta-3\gamma = \delta.$$

We now substitute equation (166) into equation (163) to get:

(167) $$w(w^3+1)\frac{d^2u}{dw^2} + [(1+\alpha+\beta)w^3 + \delta]\frac{du}{dw} + \alpha\beta w^2 u = 0.$$

Equation (167) is an equation of the [0,5,0] class which we will now solve.

We assume a solution of equation (167) to be of the form given by equation (18). Differentiating equation (18) with respect to w, gives equation (19). Differentiating equation (19) with respect to w gives equation (20). We then substitute equations (18) - (20) into equation (167) to get the following equation:

$$(168)\quad w(w^3+1)[c_0\lambda(\lambda-1)w^{\lambda-2} + c_3(\lambda+3)(\lambda+2)w^{\lambda+1} + \cdots$$
$$+ c_{3r}(\lambda+3r)(\lambda+3r-1)w^{\lambda+3r-2}$$
$$+ c_{3r+3}(\lambda+3r+3)(\lambda+3r+2)w^{\lambda+3r+1} + \cdots]$$
$$+[(1+\alpha+\beta)w^3 + \delta][c_0\lambda w^{\lambda-1} + c_3(\lambda+3)w^{\lambda+2} + \cdots$$
$$+ c_{3r}(\lambda+3r)w^{\lambda+3r-1}$$
$$+ c_{3r+3}(\lambda+3r+3)w^{\lambda+3r+2} + \cdots]$$
$$+ \alpha\beta w^2[c_0w^{\lambda} + c_3w^{\lambda+3} + \cdots + c_{3r}w^{\lambda+3r}$$
$$+ c_{3r+3}w^{\lambda+3r+3} + \cdots] = 0,$$

where we are assuming $c_0 \neq 0$.

To obtain the indicial equation, we equate the coefficient of $w^{\lambda-1}$ to zero:

$$(169)\quad c_0\lambda(\lambda-1) + c_0\lambda\delta = 0.$$

Solving equation (169), we get:

$\lambda_1 = 0$

$\lambda_2 = 1-\delta$. These are the exponents at $z = 1$.

To obtain the two term recurrence equation, we equate the coefficient of $w^{\lambda+3r+2}$ to zero, thus obtaining:

$$(170)\quad c_{3r}(\lambda+3r)(\lambda+3r-1) + c_{3r+3}(\lambda+3r+3)(\lambda+3r+2)$$
$$+ c_{3r}(\lambda+3r)(1+\alpha+\beta) + c_{3r+3}(\lambda+3r+3)\delta$$
$$+ c_{3r}(\alpha\beta) = 0.$$

Solving equation (170) for $c_{3r+3}$ we get:

$$c_{3r+3} = \frac{-[(\lambda+3r)(\lambda+3r-1) + (\lambda+3r)(1+\alpha+\beta) + \alpha\beta]c_{3r}}{(\lambda+3r+3)(\lambda+3r+2) + (\lambda+3r+3)\delta}.$$

This simplifies to the following:

$$(171)\qquad c_{3r+3} = \frac{-(\lambda+3r+\alpha)(\lambda+3r+\beta)c_{3r}}{(\lambda+3r+3)(\lambda+3r+2+\delta)}.$$

Set r = 0 in equation (171) to get:

$$(172)\qquad c_3 = \frac{-(\lambda+\alpha)(\lambda+\beta)}{(\lambda+3)(\lambda+2+\delta)}c_0.$$

Set r = 1 in equation (171) to get:

$$(173)\qquad c_6 = \frac{-(\lambda+3+\alpha)(\lambda+3+\beta)c_3}{(\lambda+3+3)(\lambda+3+2+\delta)}$$

Substituting equation (172) into (173) gives the following:

$$(174)\qquad c_6 = \frac{(-1)^2(\lambda+\alpha)(\lambda+\alpha+3)(\lambda+\beta)(\lambda+\beta+3)}{(\lambda+3)(\lambda+3+3)(\lambda+2+\delta)(\lambda+2+\delta+3)}c_0.$$

Set r = n-1 in equation (171) to get:

$$(175)\quad c_{3n} = \frac{(-1)^n(\lambda+\alpha)(\lambda+\alpha+3)\cdots(\lambda+\alpha+3n-3)(\lambda+\beta)\cdots(\lambda+\beta+3n-3)}{(\lambda+3)(\lambda+3+3)\cdots(\lambda+3+3n-3)(\lambda+2+\delta)\cdots(\lambda+2+\delta+3n-3)}c_0.$$

Divide the numerator and denominator of equation (175) by $3^{2n}$ to get the following expression:

$$(176)\quad c_{3n} = \frac{(-1)^n(\frac{\lambda+\alpha}{3})(\frac{\lambda+\alpha}{3}+1)\cdots(\frac{\lambda+\alpha}{3}+n-1)(\frac{\lambda+\beta}{3})\cdots(\frac{\lambda+\beta}{3}+n-1)}{(\frac{\lambda+3}{3})(\frac{\lambda+3}{3}+1)\cdots(\frac{\lambda+3}{3}+n-1)(\frac{\lambda+2+\delta}{3})\cdots(\frac{\lambda+2+\delta}{3}+n-1)}c_0.$$

Writing a solution of equation (167) in the form given by equation (82), we immediately have, for $\lambda = 0$;

$$(177)\quad \bar{u}_1(w) = \bar{K}_1\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n(\frac{\alpha}{3})(\frac{\alpha}{3}+1)\cdots(\frac{\alpha}{3}+n-1)(\frac{\beta}{3})\cdots(\frac{\beta}{3}+n-1)}{n!(\frac{2+\delta}{3})(\frac{2+\delta}{3}+1)\cdots(\frac{2+\delta}{3}+n-1)}w^{3n}\right].$$

We now compare equations (177) and (84), and establish the following correspondences:

$a = \alpha/3$; $c = (2+\delta)/3$;

$b = \beta/3$; $z = -w^3$. With these correspondences, we can write equation (177) in terms of hypergeometric functions, thus:

(178) $\bar{u}_1(w) = \bar{K}_1 F(\alpha/3, \beta/3; \frac{2+\delta}{3}; -w^3)$; where $\delta \neq 1 - 3n$ for $(n = 1,2,\ldots)$.

Now from equation (90) $w = z - 1$, hence equation (178) becomes:

$\bar{u}_1(z-1) = \bar{K}_1 F[\alpha/3, \beta/3; (\delta+2)/3; (1-z)^3]$, or upon letting $u_1(z) \equiv \bar{u}_1(z-1)$, we have the final result, namely:

(179) $u_1(z) = \bar{K}_1 F[\alpha/3; \beta/3; (\delta+2)/3; (1-z)^3]$.

Again in equation (176), we set $\lambda = 1 - \delta$ thus obtaining:

$$(180)\quad c_{3n} = \frac{(-1)^n(\frac{1-\delta+\alpha}{3})\cdots(\frac{1-\delta+\alpha}{3}+n-1)(\frac{1-\delta+\beta}{3})\cdots(\frac{1-\delta+\beta}{3}+n-1)}{n!(\frac{1-\delta+3}{3})(\frac{1-\delta+3}{3}+1)\cdots(\frac{1-\delta+3}{3}+n-1)}c_o.$$

Substituting into equation (82) with $\lambda = 1 - \delta$, gives:

(181) $\bar{u}_2(w) = (\bar{K}_2 w^{1-\delta})(S)$, where

$$S = 1 + \sum_{n=1}^{\infty} \frac{(\frac{1-\delta+\alpha}{3})\cdots(\frac{1-\delta+\alpha}{3}+n-1)(\frac{1-\delta+\beta}{3})\cdots(\frac{1-\delta+\beta}{3}+n-1)(-w^3)^n}{n!(\frac{1-\delta+3}{3})(\frac{1-\delta+3}{3}+1)\cdots(\frac{1-\delta+3}{3}+n-1)}.$$

Again, we may write equation (181) in terms of hypergeometric functions to get:

$$(182)\quad u_2(z) = \bar{K}_2(z-1)^{1-\delta}F[(1-\delta+\alpha)/3,\ (1-\delta+\beta)/3;\ (4-\delta)/3;\ (1-z)^3].$$

A general solution of equation (167) in a neighborhood of z = 1 is a linear combination of equations (179) and (182), thus:

$$(183)\quad u(z) = K_1F[\alpha/3,\ \beta/3;\ (\delta+2)/3;\ (1-z)^3]$$

$$+ K_2(z-1)^{1-\delta}F[\tfrac{1-\delta+\alpha}{3},\ \tfrac{1-\delta+\beta}{3};\ \tfrac{4-\delta}{3};\ (1-z)^3],$$

where $\delta \neq 1 \pm 3n$, $(n = 1,2,3,\ldots)$, and $|z-1| < 1$ for absolute convergence.

We will now apply Scheffé's criteria to equation (1), and obtain a solution to the resulting equation in a neighborhood of z = a.

Solution in a Neighborhood of z = a, where
$a \neq 0, 1, b, \text{ or } \infty$

In order to investigate the solution of equation (1) in a neighborhood of z = a, we initially make the variable transformation:

(184) $w = z - a$, or

(185) $z = w + a$.

From equation (184) it follows that

(186) $\frac{du}{dw} = \frac{du}{dz}$, and

(187) $\frac{d^2u}{dw^2} = \frac{d^2u}{dz^2}$

Substituting equations (184) - (187) into equation (1) gives us the following equation:

(188) $$(w+a)(w+a-1)(w)(w+a-b)\frac{d^2u}{dw^2} + [A_1(w+a)^3 + A_2(w+a)^2 + A_3(w+a) + A_4]\frac{du}{dw} + [B_1(w+a)^2 + B_2(w+a) + B_3]u = 0.$$

An investigation of a solution of equation (188) in a neighborhood of w = 0 is tantamount to an investigation of a solution of equation (1) in a neighborhood of z = a.

We now write equation (188) as follows:

(189) $p_2(w)\frac{d^2u}{dw^2} + p_1(w)\frac{du}{dw} + p_0(w)u = 0$, where

(190) $p_2(w) = (w+a)(w+a-1)(w)(w+a-b)$,

(191) $p_1(w) = A_1(w+a)^3 + A_2(w+a)^2 + A_3(w+a) + A_4$, and

(192) $p_0(w) = B_1(w+a)^2 + B_2(w+a) + B_3$.

We recall that $A_i$, $(i = 1,2,3,4)$, and $B_j$, $(j = 1,2,3)$, are given by equations (2) - (8).

Simplifying equations (190) - (192) in turn, we get:

$$p_2(w) = (w^2+aw)(w^2+aw-bw+aw+a^2-ab-w-a+b)$$
$$= w^4+aw^3-bw^3+aw^3+a^2w^2-abw^2.$$
$$-w^3-aw^2+bw^2+aw^3+a^2w^2-abw^2$$
$$+a^2w^2+a^3w-a^2bw-aw^2-a^2w+abw.$$
$$p_2(w) = w^4+w^3(a-b+a-1+a)$$
$$+w^2(a^2-ab-a+b+a^2-ab+a^2-a)$$
$$+w(a^3-a^2b-a^2+ab).$$

Finally we get:

$$(193) \quad p_2(w) = w^4+w^3(3a-b-1) + w^2[(2a-1)(2a-b)-a^2]$$
$$+w(a)(a-1)(a-b).$$

Similarly we may write $p_1(w)$ as follows:

$$p_1(w) = A_1(w^3+3w^2a+3wa^2+a^3) + A_2(w^2+2aw+a^2)$$
$$+A_3(w+a) + A_4, \text{ or finally,}$$
$$(194) \quad p_1(w) = A_1w^3+w^2(3aA_1+A_2) + w(3a^2A_1+2aA_2+A_3)$$
$$+ (A_1a^3+A_2a^2+A_3a+A_4).$$

Similarly, $p_o(w)$ becomes:

$$(195) \quad p_o(w) = B_1w^2 + w(2aB_1+B_2) + (a^2B_1+aB_2+B_3).$$

Substituting equations (193) - (195) into equation (189) gives us the differential equation in the desired form:

$$(196)\quad \{w^4 + w^3(3a-b-1) + w^2[(2a-1)(2a-b)-a^2] + w(a)(a-1)(a-b)\}\frac{d^2u}{dw^2}$$
$$+ [A_1w^3 + w^2(3aA_1+A_2) + w(3a^2A_1+2aA_2+A_3) + (A_1a^3+A_2a^2+A_3a+A_4)]\frac{du}{dw}$$
$$+ [B_1w^2 + w(2aB_1+B_2) + (a^2B_1+aB_2+B_3)]u = 0.$$

Equation (196) is the equation to which we wish to apply Scheffé's criteria.

We will use Scheffé's equation in the form given by equation (104), and we initially let $j = 2$, thus obtaining:

$$(197)\quad p_2(w) = S_2w^{2-m} - T_2w^{h+2-m}.$$

At this point we wish to satisfy the restriction given by equation (14). To accomplish this, we equate the right member of equation (197) to the right member of equation (193), thus obtaining the following identity:

$$(198)\quad S_2w^{2-m} - T_2w^{h+2-m} \equiv w^4 + w^3(3a-b-1) + w^2[(2a-1)(2a-b)-a^2] + w(a)(a-1)(a-b).$$

We initially assume $T_2 = 0$, and then equate coefficients of like powers of w in both members of identity (198) to get the following set of equations:

(i) $S_2 = 1$, hence $2 - m = 4$,

(ii) $3a-b-1 = 0$,

(iii) $(2a-1)(2a-b)-a^2 = 0$, and

(iv) $a(a-1)(a-b) = 0$.

Any attempt to solve equations (ii) - (iv) simultaneously leads to an inconsistency; therefore, we conclude that $T_2 \neq 0$. Similarly, if we assume $S_2 = 0$ in identity (198), we are led to the same set of inconsistent equations. Consequently we conclude that $S_2 \neq 0$. The above analysis implies $S_2T_2 \neq 0$, and the restriction given by equation (14) is satisfied.

One restriction in Scheffé's criteria is that h be a positive integer. Therefore, in identity (198) $h + 2 - m > 2 - m$, and we conclude that $h + 2 - m = 4$, or $h = 2 + m$. We can now deduce the following cases to be investigated:

Case I: $h = 1$; $m = -1$

Case II: $h = 2$; $m = 0$

Case III: $h = 3$; $m = 1$.

That the above cases form an exhaustive set, may be demonstrated as follows: Let $h > 3$, then $m \geq 2$, and from identity (198) it follows that $S_2 = 0$, contrary to previous observations. Also since h is a positive integer, then $h \geq 1$, consequently the above set is exhaustive. We will now study each of the above cases in turn.

Case I: h = 1; m = -1

Letting j = 2, h = 1, and m = -1 in equation (104) gives us the following expression:

(199) $p_2(w) = S_2w^3 - T_2w^4$.

Using equation (199) in conjunction with equation (193) yields the following identity:

(200) $p_2(w) = S_2w^3 - T_2w^4 \equiv w^4 + w^3(3a-b-1) + w^2[(2a-1)(2a-b)-a^2] + w(a)(a-1)(a-b)$.

Equating coefficients of like powers of w in both members of identity (200), gives the following equations:

(i) $T_2 = -1$,

(ii) $S_2 = 3a-b-1$,

(iii) $(2a-1)(2a-b)-a^2 = 0$, and

(iv) $a(a-1)(a-b) = 0$.

From equations (ii) - (iv) we deduce the following values for a, b, and $S_2$:

a = 0; b = 0; $S_2 = -1$;

a = 1; b = 1; $S_2 = +1$; and

a = b; b = 0,1; $S_2 = \overline{+}1$.

From this set of values, we choose a = b = 0, hence $S_2 = -1$. Substituting this information into equation (199) gives the following equation:

(201) $p_2(w) = w^4 - w^3$.

In order to compute $p_1(w)$, we set $j = 1$, $h = 1$, and $m = -1$ in equation (104). We then equate the result to equation (194) to get the following identity:

$$(202)\quad p_1(w) = S_1w^2 - T_1w^3 \equiv A_1w^3 + w^2(3aA_1+A_2) + w(3a^2A_1+2aA_2+A_3) + (A_1a^3+A_2a^2+A_3a+A_4).$$

Equating coefficients of like powers of w in both members of identity (202) yields the following set of equations:

$$(i)\quad T_1 = -A_1,$$
$$(ii)\quad S_1 = 3aA_1+A_2,$$
$$(iii)\quad 3a^2A_1+2aA_2+A_3 = 0, \text{ and}$$
$$(iv)\quad A_1a^3+A_2a^2+A_3a+A_4 = 0.$$

Using the fact that $a = b = 0$, we have from equation (3) $S_1 = -(1+\alpha+\beta) + \delta$. Also from equation (2) we have $T_1 = -(1+\alpha+\beta)$. Equation (202) now becomes:

$$(203)\quad p_1(w) = (1+\alpha+\beta)w^3 - (1+\alpha+\beta-\delta)w^2.$$

The computation of $p_o(w)$ proceeds in a similar fashion to the above. That is, we let $j = 0$, $h = 1$, and $m = -1$ in equation (104), then equate the result to the right member of equation (195) to obtain the following identity:

$$(204)\quad p_o(w) = S_ow - T_ow^2 \equiv B_1w^2 + w(2aB_1+B_2) + (a^2B_1+aB_2+B_3).$$

Again we equate coefficients of like powers of w in both members of identity (204) to obtain the following equations:

(i) $T_o = -B_1$,

(ii) $S_o = 2aB_1+B_2$, and

(iii) $a^2B_1+aB_2+B_3 = 0$.

Now from equations (6) - (8), and from the restriction a = b = 0, we deduce the following equations: $T_o = -\alpha\beta$, $S_o = -\alpha\beta p$. The value for $S_o$ is deduced as follows: Since a = 0, then from equation (iii) above, $B_3 = 0$ which implies, by way of equation (8), that p = 0, or q = 0, or p = q = 0. $\alpha\beta \neq 0$, because if $\alpha\beta = 0$, then equation (1) is essentially a first order differential equation. We choose q = 0. Then $S_o = -\alpha\beta p$.

Substituting the above values of $T_o$ and $S_o$ into equation (204) gives us the following equation:

(205) $p_o(w) = \alpha\beta w^2 - \alpha\beta pw$.

We now substitute equations (201), (203), and (205) into equation (189) to obtain the following differential equation:

$$(206)\quad (w^4 - w^3)\frac{d^2u}{dw^2} + [(1+\alpha+\beta)w^3 - (1+\alpha+\beta-\delta)w^2]\frac{du}{dw} + \alpha\beta w(w-p)u = 0.$$

Division of both members of equation (206) by w gives us the equation in the desired form:

$$(207)\qquad w^2(w-1)\frac{d^2u}{dw^2} + [(1+\alpha+\beta)w^2 - (1+\alpha+\beta-\delta)w]\frac{du}{dw}$$

$$+ \alpha\beta(w-p)u = 0.$$

In the derivation of equation (207), we chose $a = b = q = 0$. Other choices of a and b will give other confluent differential equations. We will now classify equation (207) in the notation of Ince.

We begin the classification by initially writing equation (207) in the form given by equation (117), thus:

$$\frac{d^2u}{dw^2} + p(w)\frac{du}{dw} + q(w)u = 0, \text{ where}$$

$$(208)\qquad p(w) = \frac{(1+\alpha+\beta)}{w-1} - \frac{(1+\alpha+\beta-\delta)}{w(w-1)}, \text{ and}$$

$$(209)\qquad q(w) = \frac{\alpha\beta(w-p)}{w^2(w-1)}.$$

From the criteria outlined in Chapter I, if $(w-c)p(w)$ and $(w-c)^2q(w)$ are both analytic at some finite point c, then c is a regular singular point of the differential equation under consideration. Hence from equations (208) and (209), we have

$$wp(w) = \frac{w(1+\alpha+\beta)}{w-1} - \frac{(1+\alpha+\beta-\delta)}{w-1}, \text{ and}$$

$$w^2q(w) = \frac{\alpha\beta(w-p)}{w-1}.$$

Since $wp(w)$ and $w^2q(w)$ are both analytic at $w = 0$, we conclude that $w = 0$ is a regular singular point of equation (207).

Again from equations (208) and (209) we have the following:

$$(210) \quad (w-1)p(w) = (1+\alpha+\beta) - \frac{(1+\alpha+\beta-\delta)}{w}, \text{ and}$$

$$(211) \quad (w-1)^2q(w) = \frac{\alpha\beta(w-1)(w-p)}{w^2}.$$

Also from equations (210) and (211) we note that $(w-1)p(w)$ and $(w-1)^2q(w)$ are both analytic at $w = 1$, hence we conclude that $w = 1$ is a regular singular point of equation (207).

In order to examine whether or not infinity is a regular singular point of equation (207), we initially make the variable transformation given by equation (124). Applying this transformation to equation (117) gives us:

$$(212) \quad \frac{d^2u}{dv^2} + [2/v - (1/v^2)p(1/v)]\frac{du}{dv} + (1/v^4)q(1/v)u = 0.$$

We now examine whether or not $v = 0$ is a regular singular point of equation (212). Thus we investigate analyticity of $v[2/v - (1/v^2)p(1/v)]$ and $v^2(1/v^4)q(1/v)$ at $v = 0$. From equations (207) and (212) we get:

$$v[2/v - (1/v^2)p(1/v)] = 2 - (1/v)\left[\frac{v(1+\alpha+\beta)}{1-v} - \frac{v^2(1+\alpha+\beta-\delta)}{1-v}\right],$$

or

$$(213)\quad [2 - (1/v)p(1/v)] = 2 - \frac{(1+\alpha+\beta)}{1 - v} + \frac{v(1+\alpha+\beta-\delta)}{1 - v}, \text{ and}$$

$$v^2(1/v^4)q(1/v) = (1/v^2)\left[\frac{v^2\alpha\beta(1-pv)}{1 - v}\right], \text{ or}$$

$$(214)\quad (1/v^2)q(1/v) = \frac{\alpha\beta(1-pv)}{1 - v}.$$

We immediately see from equations (213) and (214) that $[2 - (1/v)p(1/v)]$ and $(1/v^2)q(1/v)$ are both analytic at $v = 0$. Consequently $v = 0$ is a regular singular point of equation (212). This implies, by way of equation (124), that infinity is a regular singular point of equation (207).

The investigation to this point shows that $w = 0$, 1, and infinity are regular singular points of equation (207). We will now determine whether or not these three points are elementary singular points. This will be accomplished by assuming a series solution of equation (207), expanding in a neighborhood of each regular singular point in turn, and then examining the difference of the roots of the respective indicial equations.

We assume a solution of equation (207) in a neighborhood of $w = 0$ to be:

$$(215)\quad u(w) = c_0w^{\lambda} + c_1w^{\lambda+1} + \cdots + c_1w^{\lambda+r} + \cdots ; c_0 \neq 0.$$

Differentiating equation (215) with respect to w gives us the following equation:

(216) $u'(w) = c_o\lambda w^{\lambda-1} + c_1(\lambda+1)w^{\lambda} + \cdots + c_r(\lambda+r)w^{\lambda+r-1} + \cdots$.

Differentiating equation (216) with respect to w gives us the following equation:

(217) $u''(w) = c_o\lambda(\lambda-1)w^{\lambda-2} + c_1(\lambda+1)(\lambda)w^{\lambda-1} + \cdots$

$$+ c_r(\lambda+r)(\lambda+r-1)w^{\lambda+r-2} + \cdots .$$

Substituting equations (215) - (217) into equation (207), and equating the coefficient of $w^{\lambda}$ to zero, gives us the following indicial equation:

(218) $c_o\lambda(\lambda-1) - c_o\lambda(1+\alpha+\beta-\delta) - c_o\alpha\beta p = 0.$

Equation (218) simplifies to:

(219) $\lambda^2 - \lambda(2+\alpha+\beta-\delta) - \alpha\beta p = 0.$

Solving equation (218) for $\lambda$ yields:

$$\lambda_1 = [(2+\alpha+\beta-\delta) - R]/2, \text{ and}$$

$$\lambda_2 = [(2+\alpha+\beta-\delta) + R]/2, \text{ where}$$

$$R = \sqrt{(2+\alpha+\beta-\delta)^2 + 4\alpha\beta p}.$$

From the above equations we see that $\lambda_2 - \lambda_1 = R \neq 1/2$, and thus w = 0 is not an elementary singular point.

We next investigate whether or not w = 1 is an elementary singular point. First we make the variable transformation v = w - 1. Equation (207) then becomes:

(220) $v(v^2+2v+1)\frac{d^2u}{dv^2} + [v^2(1+\alpha+\beta) + v(1+\alpha+\beta+\delta) + \delta]\frac{du}{dv}$

$$+ \alpha\beta(v+1-p)u = 0.$$

We assume a solution of equation (220), in a neighborhood of $v = 0$, that is $w = 1$, to be of the form:

$$(221)\quad u(v) = c_0 v^{\lambda} + c_1 v^{\lambda+1} + \cdots + c_r v^{\lambda+r} + \cdots ; \; c_0 \neq 0.$$

Differentiating equation (221) with respect to v gives us the following equation:

$$(222)\quad u'(v) = c_0 \lambda v^{\lambda-1} + c_1(\lambda+1)v^{\lambda} + \cdots + c_r(\lambda+r)v^{\lambda+r-1} + \cdots ,$$

and differentiating equation (222) with respect to v yields:

$$(223)\quad u''(v) = c_0(\lambda)(\lambda-1)v^{\lambda-2} + c_1(\lambda+1)(\lambda)^{\lambda-1} + \cdots + c_r(\lambda+r)(\lambda+r-1)v^{\lambda+r-2} + \cdots .$$

Substituting equations (221) - (223) into equation (220) and equating the coefficient of $v^{\lambda-1}$ to zero, yields the following indicial equation:

$$(224)\quad c_0\lambda(\lambda-1) + c_0\lambda\delta = 0.$$

Solving equation (224) for $\lambda$ gives us $\lambda_1 = 0$, and $\lambda_2 = 1 - \delta$, and hence $\lambda_2 - \lambda_1 \neq 1/2$. We therefore conclude that $v = 0$ is not an elementary singular point of equation (220). This implies $w = 1$ is not an elementary singular point of equation (207).

Finally, we investigate whether or not infinity is an elementary singular point of equation (207). First, however, we make the variable transformation $v = 1/w$, and apply this transformation to equation (207), thus obtaining equation (212), where

(225) $[2/v - (1/v^2)p(1/v)] = 2/v - (1/v^2)\left[\frac{v(1+\alpha+\beta)}{1-v} - \frac{v^2(1+\alpha+\beta-\delta)}{1-v}\right]$,

and

(226) $(1/v^4)q(1/v) = (1/v^4)[v^2\alpha\beta(1-pv)/(1-v)]$.

Substituting equations (225) and (226) into equation (212) we get:

(227) $$\frac{d^2u}{dv^2} + \left[2/v - \frac{1+\alpha+\beta}{v(1-v)} + \frac{1+\alpha+\beta-\delta}{1-v}\right]\frac{du}{dv}$$

$$+ [\alpha\beta(1-pv)/v^2(1-v)]u = 0.$$

Equation (227) immediately reduces to:

(228) $$v^2(v-1)\frac{d^2u}{dv^2} + [2v(v-1) + (1+\alpha+\beta)v - (1+\alpha+\beta-\delta)v^2]\frac{du}{dv}$$

$$+ \alpha\beta(pv-1)u = 0.$$

Substituting equations (221) - (223) into equation (228), then equating the coefficient of $v^\lambda$ to zero gives us the following indicial equation:

(229) $-c_o\lambda(\lambda-1) + c_o\lambda(-2+1+\alpha+\beta) - c_o\alpha\beta = 0.$

Equation (229) reduces to

(230) $(\lambda-\alpha)(\lambda-\beta) = 0$, so that $\lambda_1 = \alpha$, and $\lambda_2 = \beta$, hence $\lambda_2 - \lambda_1 \neq 1/2$. This implies $v = 0$ is not an elementary singular point of equation (228), which, by way of the transformation $w = 1/v$, implies that infinity is not an elementary singular point of equation (207).

Inasmuch as equation (207) has no elementary singular points, three regular singular points, and no irregular singular points, we classify it in the notation of Ince as an equation of the [0,3,0] class.

We will now study Case II.

Case II: h = 2; m = 0

We now let j = 2, h = 2, and m = 0 in equation (104), then equate the result to the right member of equation (193) to obtain the following identity:

$$(231)\quad p_2(w) = S_2w^2 - T_2w^4 \equiv w^4 + w^3(3a-b-1) + w^2[(2a-1)(2a-b)-a^2] + w(a)(a-1)(a-b).$$

Equating coefficients of like powers of w in both members of the above identity gives the following set of equations:

(i) $T_2 = -1$,

(ii) $S_2 = (2a-1)(2a-b)-a^2$,

(iii) $3a-b-1 = 0$, and

(iv) $a(a-1)(a-b) = 0$.

Solving equations (iii) and (iv) simultaneously gives us the following possibilities for a and b:

$a = 0$; $b = -1$,

$a = 1$; $b = 2$, and

$a = b$; $b = 1/2$.

Of the above choices, we choose $a = 0$, $b = -1$, so that $T_2 = -1$, and $S_2 = -1$. Equation (231) then becomes:

$$(232)\quad p_2(w) = w^4 - w^2.$$

Other choices of a and b will result in other expressions for $p_2(w)$.

Again in equation (104) we let j = 1, h = 2, and m = 0, then equate the result to the right member of equation (194), thus obtaining the following identity:

$$(233)\quad p_1(w) = S_1w - T_1w^3 \equiv A_1w^3 + w^2(3aA_1+A_2) + w(3a^2A_1+2aA_2+A_3) + (A_1a^3 + A_2a^2+A_3a+A_4).$$

Equating coefficients of like powers of w in both members of identity (233), and remembering that a = 0, gives us the following set of equations:

(i) $T_1 = -A_1$,

(ii) $S_1 = A_3$,

(iii) $A_2 = 0$, and

(iv) $A_4 = 0$.

Now from equations (2) - (5), we have for a = 0 and b = -1, $A_3 = -(\gamma+\epsilon)$, and $-A_1 = -(1+\alpha+\beta)$. Also $A_2 = 0$ implies $\eta = \delta$. Substituting the above information into equation (233) gives the following for $p_1(w)$:

$$(234)\quad p_1(w) = (1+\alpha+\beta)w^3 - (\gamma+\epsilon)w.$$

Finally, in equation (104) we let j = 0, h = 2, and m = 0, then equate the result to the right member of equation (195) to get the following identity:

$$(235)\quad p_0(w) = S_0 - T_0w^2 \equiv B_1w^2 + w(2aB_1+B_2) + (a^2B_1+aB_2+B_3).$$

From this identity we can obtain the following set of equations immediately:

(i) $T_o = -B_1$,

(ii) $S_o = a^2B_1+aB_2+B_3$, and

(iii) $2aB_1+B_2 = 0$.

Comparing the above equations with equations (6) - (8), and remembering that $a = 0$, we get the following useful relationships:

(i) $T_o = -\alpha\beta$,

(ii) $S_o = -\alpha\beta p^2$, and

(iii) $p = -q$, since $\alpha\beta \neq 0$.

Substituting the above information into equation (235) gives us the following expression for $p_o(w)$:

(236) $p_o(w) = \alpha\beta w^2 - \alpha\beta p^2$.

We now substitute equations (232), (234), and (236) into equation (189), thus obtaining the following differential equation:

$$(237)\quad (w^4-w^2)\frac{d^2u}{dw^2} + [(1+\alpha+\beta)w^3 - (\gamma+\epsilon)w]\frac{du}{dw} + \alpha\beta(w^2-p^2)u = 0.$$

Equation (237) was derived by choosing $a = 0$, $b = 1$, and $p = -q$. Other choices will result in other confluent differential equations. Equation (237) may be classified in a manner similar to that used to classify equation (207).

We will now investigate Case III.

Case III: $h = 3$; $m = 1$

We begin the investigation of Case III by letting $j = 2$, $h = 3$, and $m = 1$ in equation (104), then equate the result to the right member of equation (193) to get the following identity:

$$(238)\quad p_2(w) = S_2w - T_2w^4 \equiv w^4 + w^3(3a-b-1) + w^2[(2a-1)(2a-b)-a^2] + w(a)(a-1)(a-b).$$

We now equate coefficients of like powers of w in both members of the above identity to obtain the following set of equations:

(i) $T_2 = -1$,

(ii) $S_2 = (a)(a-1)(a-b)$,

(iii) $3a-b-1 = 0$, and

(iv) $(2a-1)(2a-b)-a^2 = 0$.

Solving equations (iii) and (iv) simultaneously gives us the following sequence of equations:

$a = \frac{1+b}{3}$ from equation (iii), and

$4a^2 - 2ab - 2a + b - a^2 = 0$ from equation (iv).

Combining the above equations and simplifying gives us the following equation:

$b^2 - b + 1 = 0$, and solving gives us

$b_1 = -\omega^2$; $b_2 = -\omega$, where 1, $\omega$, and $\omega^2$ are the three cube roots of unity. Corresponding to these values of b, we note that

$$a_1 = \frac{1-\omega^2}{3}; \text{ and } a_2 = \frac{1-\omega}{3}.$$

We will choose $a_1 = (1-\omega^2)/3$ and $b_1 = -\omega^2$ for future discussion. From these values of a and b, we deduce the following useful relationships:

$$a_1 b_1 = \frac{\omega-\omega^2}{3} = \frac{i\sqrt{3}}{3}, \text{ where } i^2 = -1,$$

$$a_1^3 = \frac{\omega-\omega^2}{9} = \frac{i\sqrt{3}}{9}, \text{ and } 3a_1^3 = a_1 b_1.$$

Now from equation (11) we have

$$S_2 = a(a^2-ab-a+b) = a^3 - a^2b - a^2 + ab.$$

By the preceding relationships, this reduces to

$$S_2 = \frac{a_1 b_1}{3} = \frac{i\sqrt{3}}{9}, \text{ and equation (238) now becomes:}$$

(239) $\quad p_2(w) = w^4 + (i\sqrt{3}/9)w.$

Again in equation (104) we let j = 1, h = 3, and m = 1, then equate the result to the right member of equation (194), thus obtaining the following identity:

$$(240)\quad p_1(w) = S_1 - T_1w^3 \equiv A_1w^3 + w^2(3aA_1+A_2) + w(3a^2A_1+2aA_2+A_3) + (A_1a^3+A_2a^2+A_3a+A_4).$$

Again we equate the coefficients of like powers of w in both members of the above identity to obtain the following equations:

(i) $T_1 = -A_1$,

(ii) $S_1 = A_1a^3+A_2a^2+A_3a+A_4$,

(iii) $3aA_1+A_2 = 0$, and

(iv) $3a^2A_1+2aA_2+A_3 = 0$.

From equations (iii) and (iv), we immediately deduce that $aA_2 + A_3 = 0$, and hence $S_1 = A_1a^3 + A_4$. This, together with equations (2) and (5), and remembering that $a_1^3 = (i\sqrt{3}/9)$, $a_1b_1 = (i\sqrt{3}/3)$, lead us to the conclusion that $T_1 = -(1+\alpha+\beta)$ and $S_1 = (i\sqrt{3}/9)(1+\alpha+\beta) - (i\sqrt{3}/3)\gamma$. Equation (240) now becomes:

$$(241)\quad p_1(w) = (1+\alpha+\beta)w^3 + [(i\sqrt{3}/9)(1+\alpha+\beta) - (i\sqrt{3}/3)\gamma].$$

Finally, we let $j = 0$, $h = 3$, and $m = 1$ in equation (104), then equate the result to the right member of equation (195) to obtain the following identity:

$$(242)\quad p_0(w) = S_0w^{-1} - T_0w^2 \equiv B_1w^2 + w(2aB_1+B_2) + (a^2B_1+aB_2+B_3).$$

Equating coefficients of like powers of w in both members of the above identity, yields the following set of equations:

(i) $T_o = -B_1$,

(ii) $S_o = 0$,

(iii) $2aB_1+B_2 = 0$, and

(iv) $a^2B_1+aB_2+B_3 = 0$.

From equations (iii) and (iv) we immediately deduce that $a^2B_1 = B_3$. Comparing the above equations with equations (6) - (8), we observe that $T_o = -\alpha\beta$, hence equation (242) becomes:

(243) $p_o(w) = \alpha\beta w^2$.

Substituting equations (239), (241), and (243) into equation (189) gives us the desired differential equation:

$$(244)\quad [w^4+(i\sqrt{3}/9)w]\frac{d^2u}{dw^2} +[(1+\alpha+\beta)w^3 + (i\sqrt{3}/9)(1+\alpha+\beta-3\gamma)]\frac{du}{dw} + \alpha\beta w^2u = 0.$$

Subject to the restrictions given in the derivation of equation (244), we will now show that $1 + \alpha + \beta - 3\gamma = \epsilon$. Thus in the derivation of $S_1$, we noted that $a_1A_2 + A_3 = 0$, hence from equations (3) and (4) we have:

$$(245)\quad -a_1[(1+\alpha+\beta)(1+a_1+b_1) - a_1\epsilon - b_1\eta - \delta] = [a_1b_1(\gamma+\epsilon) + a_1(\gamma+\eta) + b_1(\gamma+\epsilon)].$$

We now substitute $a_1 = (1+b_1)/3$ into equation (245), then divide both members of the result by $(-a_1)$ to get:

$$(1+\alpha+\beta)[1+(1+b_1)/3 + b_1] - \epsilon(1+b_1)/3 - b_1\eta - \delta = b_1(\gamma+\delta) + (\gamma+\eta) + (\gamma+\epsilon)(b_1/a_1), \text{ or}$$

$$[(1+\alpha+\beta)(4+4b_1)/3] - \epsilon[(1+b_1)/3] - \eta(1+b_1) - \gamma(1+b_1) - \delta(1+b_1) = (\gamma+\epsilon)(b_1/a_1).$$

This may be written:

$$(3a_1)(1+\alpha+\beta-\epsilon-\eta-\gamma-\delta) + a(1+\alpha+\beta)$$
$$= -2a\epsilon + (\gamma+\epsilon)(b_1/a_1).$$

From equation (12) in Chapter V, we noted that

$1 + \alpha + \beta - \gamma - \delta - \eta - \epsilon = 0$, hence we multiply both members of the above equation by $a_1^2$, and reduce it as follows:

$$a_1^3(1+\alpha+\beta) - a_1b_1\gamma = \epsilon(a_1b_1-2a_1^3).$$

But again we recall that $3a_1^3 = a_1b_1$, hence the above equation reduces to the following:

$$(1+\alpha+\beta) - \frac{a_1b_1}{a_1^3}\gamma = \epsilon(a_1b_1/a_1^3-2), \text{ or}$$

$1 + \alpha + \beta - 3\gamma = \epsilon$, which is the desired result. Substitution of this expression into equation (244) gives us the final result:

$$(246) \quad [w^4+(i\sqrt{3}/9)w]\frac{d^2u}{dw^2} + [(1+\alpha+\beta)w^3 + (i\sqrt{3}/9)\epsilon]\frac{du}{dw}$$
$$+ \alpha\beta w^2u = 0.$$

Equation (246) was derived by choosing $a = (1-\omega^2)/3$, $b = -\omega^2$, $h = 3$, and $m = 1$. Other choices for a and b will determine other non-confluent differential equations.

We will now solve equation (246) in a neighborhood of $w = 0$, that is, in a neighborhood of $z = a$.

We assume a solution of equation (246) to be of the form given by equation (18). We then differentiate equation (18) with respect to w to get equation (19). Following this, we differentiate equation (19) to get equation (20). We then substitute equations (18) - (20) into equation (246) to obtain the following result:

$$
\begin{aligned}
(247)\quad & [w^4 + (i\sqrt{3}/9)w][c_0\lambda(\lambda-1)w^{\lambda-2} + c_3(\lambda+3)(\lambda+2)w^{\lambda+1} + \cdots \\
& \qquad + c_{3r}(\lambda+3r)(\lambda+3r-1)w^{\lambda+3r-2} \\
& \qquad + c_{3r+3}(\lambda+3r+3)(\lambda+3r+2)w^{\lambda+3r+1} + \cdots] \\
& + [(1+\alpha+\beta)w^3 + (i\sqrt{3}/9)\epsilon][c_0\lambda w^{\lambda-1} + c_3(\lambda+3)w^{\lambda+2} + \cdots \\
& \qquad + c_{3r}(\lambda+3r)w^{\lambda+3r-1} \\
& \qquad + c_{3r+3}(\lambda+3r+3)w^{\lambda+3r+2} + \cdots] \\
& + [\alpha\beta w^2][c_0 w^{\lambda} + c_3 w^{\lambda+3} + \cdots + c_{3r}w^{\lambda+3r} \\
& \qquad + c_{3r+3}w^{\lambda+3r+3} + \cdots] = 0.
\end{aligned}
$$

We assume that $c_0 \neq 0$ in the above equation.

In order to obtain the indicial equation, we equate the coefficient of $w^{\lambda-1}$ to zero as follows:

$$(248)\quad c_0\lambda(\lambda-1)(i\sqrt{3}/9) + c_0\lambda(i\sqrt{3}/9)\epsilon = 0.$$

Solving equation (248) for $\lambda$ yields:

$\lambda = 0, 1 - \epsilon$ . This verifies our choice of exponents at $z = a$.

We can get the two term recurrence equation by equating the coefficient of $w^{\lambda+3r+2}$ to zero, thus obtaining:

(249) $$c_{3r}(\lambda+3r)(\lambda+3r-1) + c_{3r+3}(\lambda+3r+3)(\lambda+3r+2)(i\sqrt{3}/9)$$
$$+ c_{3r}(\lambda+3r)(1+\alpha+\beta) + c_{3r+3}(\lambda+3r+3)(i\sqrt{3}/9)\epsilon$$
$$+ c_{3r}\alpha\beta = 0.$$

Solving equation (249) for $c_{3r+3}$ gives us the following equation:

(250) $$c_{3r+3} = \frac{-[(\lambda+3r)(\lambda+3r-1)+(\lambda+3r)(1+\alpha+\beta)+\alpha\beta]c_{3r}}{(\lambda+3r+3)(i\sqrt{3}/9)(\lambda+3r+2+\epsilon)}.$$

Equation (250) is easily simplified to the following form:

(251) $$c_{3r+3} = \frac{(3i\sqrt{3})(\lambda+3r+\alpha)(\lambda+3r+\beta)}{(\lambda+3r+3)(\lambda+3r+2+\epsilon)}c_{3r}.$$

Set r = 0 in equation (251) to obtain:

(252) $$c_3 = \frac{(3i\sqrt{3})(\lambda+\alpha)(\lambda+\alpha+\beta)}{(\lambda+3)(\lambda+2+\epsilon)}c_o$$

Now let r = 1 in equation (251) to get:

(253) $c_6 = \frac{(3i\sqrt{3})(\lambda+3+\alpha)(\lambda+3+\beta)}{(\lambda+3+3)(\lambda+3+2+\epsilon)}c_3$, and substitute equation (252) into equation (253) to get $c_6$ in terms of $c_o$, thus:

(254) $$c_6 = \frac{(3i\sqrt{3})^2(\lambda+\alpha)(\lambda+\alpha+3)(\lambda+\beta)(\lambda+\beta+3)}{(\lambda+3)(\lambda+3+3)(\lambda+2+\epsilon)(\lambda+2+\epsilon+3)}c_o.$$

We can now generalize by letting r = n - 1 in equation (251) thus obtaining:

$$(255)\quad c_{3n} = \frac{(3i\sqrt{3})^n(\lambda+\alpha)\cdots(\lambda+\alpha+3n-3)(\lambda+\beta)\cdots(\lambda+\beta+3n-3)}{(\lambda+3)(\lambda+3+3)\cdots(\lambda+3+3n-3)(\lambda+2+\epsilon)\cdots(\lambda+2+\epsilon+3n-3)}c_o.$$

Dividing numerator and denominator of equation (255) by $3^{2n}$ we get the following expression:

$$(256)\quad c_{3n} = \frac{(3i\sqrt{3})^n(\frac{\lambda+\alpha}{3})\cdots(\frac{\lambda+\alpha}{3}+n-1)(\frac{\lambda+\beta}{3})\cdots(\frac{\lambda+\beta}{3}+n-1)}{(\frac{\lambda+3}{3})(\frac{\lambda+3}{3}+1)\cdots(\frac{\lambda+3}{3}+n-1)(\frac{\lambda+2+\epsilon}{3})\cdots(\frac{\lambda+2+\epsilon}{3}+n-1)}c_o.$$

We can now write a solution of equation (246) in the form given by equation (82) repeated here for convenience, thus:

$\bar{u}(w) = c_o w^\lambda\left[1 + \sum_{n=1}^{\infty}(c_{3n}/c_o)w^{3n}\right]$, where $c_{3n}$ is given by equation (256). We now have, for $\lambda = 0$, the following solution:

$$(257)\quad \bar{u}_1(w) = \bar{K}_1\left[1 + \sum_{n=1}^{\infty}\frac{(3i\sqrt{3})^n(\frac{\alpha}{3})\cdots(\frac{\alpha}{3}+n-1)(\frac{\beta}{3})\cdots(\frac{\beta}{3}+n-1)}{n!(\frac{2+\epsilon}{3})(\frac{2+\epsilon}{3}+1)\cdots(\frac{2+\epsilon}{3}+n-1)}w^{3n}\right].$$

Comparing the above series with equation (84), we see that we can express the above solution in terms of the hypergeometric function. First, however, we set up the following correspondences:

$a = \alpha/3$; $\quad c = (2+\epsilon)/3$, and

$b = \beta/3$; $\quad z = 3i\sqrt{3}w^3$.

In terms of the hypergeometric function, equation (257) now becomes:

$$(258)\quad \bar{u}_1(w) = \bar{K}_1 F(\tfrac{\alpha}{3}, \tfrac{\beta}{3}; \tfrac{2+\epsilon}{3}; 3i\sqrt{3}w^3).$$

We finally recall that $w = z - a$, and hence we let

$\bar{u}_1(w) = \bar{u}_1(z-a) = u_1(z)$, so that

$$(259)\quad u_1(z) = \bar{K}_1 F[\alpha/3,\ \beta/3;\ (2+\epsilon)/3;\ 3i\sqrt{3}(z-a)^3].$$

We now let $\lambda = 1 - \epsilon$, so that equation (256) becomes:

$$(260)\quad c_{3n} = \frac{(3i\sqrt{3})^n(\frac{1-\epsilon+\alpha}{3})\cdots(\frac{1-\epsilon+\alpha}{3}+n-1)(\frac{1-\epsilon+\beta}{3})\cdots(\frac{1-\epsilon+\beta}{3}+n-1)}{n!(\frac{4-\epsilon}{3})(\frac{4-\epsilon}{3}+1)\cdots(\frac{4-\epsilon}{3}+n-1)}c_0.$$

Correspondingly, equation (257) becomes:

$$(261)\quad \bar{u}_2(w) = (\bar{K}_2 w^{1-\epsilon})(S+1), \text{ where}$$

$$S = \sum_{n=1}^{\infty} \frac{(3i\sqrt{3})^n(\frac{1-\epsilon+\alpha}{3})\cdots(\frac{1-\epsilon+\alpha}{3}+n-1)(\frac{1-\epsilon+\beta}{3})\cdots(\frac{1-\epsilon+\beta}{3}+n-1)}{n!(\frac{4-\epsilon}{3})(\frac{4-\epsilon}{3}+1)\cdots(\frac{4-\epsilon}{3}+n-1)}w^{3n}.$$

Again we can express the above solution in terms of the hypergeometric function as follows:

$$(262)\quad \bar{u}_2(w) = \bar{K}_2 w^{1-\epsilon} F[\tfrac{1-\epsilon+\alpha}{3},\ \tfrac{1-\epsilon+\beta}{3};\ \tfrac{4-\epsilon}{3};\ 3i\sqrt{3}w^3].$$

Likewise, we recall again that $w = z - a$, so that

$\bar{u}_2(w) = \bar{u}_2(z-a) = u_2(z)$, and the above solution becomes:

$$(263)\quad u_2(z) = \bar{K}_2(z-a)^{1-\epsilon} F[\tfrac{1-\epsilon+\alpha}{3},\ \tfrac{1-\epsilon+\beta}{3};\ \tfrac{4-\epsilon}{3};\ 3i\sqrt{3}(z-a)^3].$$

A general solution of equation (246) in a neighborhood of w = 0, that is, in a neighborhood of z = a, is a linear combination of equations (259) and (262), or:

$$(264)\quad u(z) = K_1 F[\alpha/3,\ \beta/3;\ (2+\epsilon)/3;\ 3i\sqrt{3}(z-a)^3] + K_2(z-a)^{1-\epsilon} F\left[\frac{1-\epsilon+\alpha}{3},\ \frac{1-\epsilon+\beta}{3};\ \frac{4-\epsilon}{3};\ 3i\sqrt{3}(z-a)^3\right].$$

In equation (264) we assume that $\epsilon \neq 1 \pm 3n$, $(n = 1,2,\ldots)$. Further, by the ratio test, equation (264) is absolutely convergent for $|z-a| < \sqrt{3}/3$.

We will now apply Scheffé's criteria to equation (1) and investigate a solution of the result in a neighborhood of z = b.

Solution in a Neighborhood of $z = b$, where $b \neq 0, 1, a, \text{ or } \infty$

We will now apply Scheffé's criteria to equation (1) and consider a solution of the result in a neighborhood of $z = b$.

We first make the variable transformation:

(265) $w = z - b$, or

(266) $z = w + b$.

From these equations it follows that

(267) $\frac{du}{dw} = \frac{du}{dz}$, and

(268) $\frac{d^2u}{dw^2} = \frac{d^2u}{dz^2}$.

Substituting equations (265) - (268) into equation (1) gives us the differential equation in the following form:

$$(w+b)(w+b-1)(w+b-a)(w)\frac{d^2u}{dw^2} + [A_1(w+b)^3 + A_2(w+b)^2 + A_3(w+b) + A_4]\frac{du}{dw} + [B_1(w+b)^2 + B_2(w+b) + B_3]u = 0.$$

This equation may be written as follows:

(269) $p_2(w)\frac{d^2u}{dw^2} + p_1(w)\frac{du}{dw} + p_o(w)u = 0$, where

(270) $p_2(w) = (w+b)(w+b-1)(w+b-a)(w)$,

(271) $p_1(w) = A_1(w+b)^3 + A_2(w+b)^2 + A_3(w+b) + A_4$, and

(272) $p_o(w) = B_1(w+b)^2 + B_2(w+b) + B_3$.

We will now simplify each of equations (270) - (272) in turn.

$$p_2(w) = (w+b)(w+b-1)(w+b-a)(w)$$

$$= (w^2+bw)(w^2+bw-aw+bw+b^2-ab-w-b+a)$$

$$= w^4+bw^3-aw^3+bw^3+b^2w^2-abw^2-w^3-bw^2$$
$$+ aw^2+bw^3+b^2w^2-abw^2+b^2w^2+b^3w-ab^2w$$
$$- bw^2-b^2w+abw$$

$$= w^4+w^3(b-a+b-1+b) + w^2(b^2-ab-b+a+b^2-ab+b^2-b)$$
$$+ w(b^3-ab^2-b^2+ab), \text{ thus finally,}$$

$$(273) \quad p_2(w) = w^4+w^3(3b-a-1) + w^2[(2b-1)(2b-a)-b^2]$$
$$+ w(b)(b-1)(b-a).$$

In a similar fashion we obtain:

$$p_1(w) = A_1(w^3+3w^2b+3wb^2+b^3) + A_2(w^2+2bw+b^2)$$
$$+ A_3(w+b) + A_4, \text{ or finally,}$$

$$(274) \quad p_1(w) = A_1w^3 + w^2(3A_1b+A_2) + w(3A_1b^2+2A_2b+A_3)$$
$$+ (A_1b^3+A_2b^2+A_3b+A_4).$$

Similarly,

$$(275) \quad p_0(w) = B_1w^2 + w(2B_1b+B_2) + (B_1b^2+B_2b+B_3).$$

Substituting equations (273) - (275) into equation (269) gives us the differential equation in the following desirable form:

$$(276)\quad \{w^4 + w^3(3b-a-1) + w^2[(2b-1)(2b-a)-b^2] + w(b)(b-1)(b-a)\}\frac{d^2u}{dw^2}$$
$$+ [A_1w^3 + w^2(3A_1b+A_2) + w(3A_1b^2+2A_2b+A_3) + (A_1b^3+A_2b^2+A_3b+A_4)]\frac{du}{dw}$$
$$+ [B_1w^2 + w(2bB_1+ B_2) + (b^2B_1+bB_2+B_3)]u = 0.$$

We reiterate that $A_i$, $(i = 1,2,3,4)$ and $B_j$, $(j = 1,2,3)$ are given by equations (2) - (8).

We now apply Scheffé's criteria to equation (276) in the form given by equation (104) since we are interested in a neighborhood of $w = 0$, that is, $z = b$. Thus in equation (104) we let $j = 2$, and equate the right member of the result to the right member of equation (273), thus obtaining the following identity:

$$(277)\quad p_2(w) = S_2w^{2-m} - T_2w^{h+2-m} \equiv w^4 + w^3(3b-a-1) + w^2[(2b-1)(2b-a)-b^2] + w(b)(b-1)(b-a).$$

In the above identity, we assume $T_2 = 0$, then equate coefficients of like powers of w in the result to obtain the following set of equations:

(i) $S_2 = 1$, since of necessity $2 - m = 4$,

(ii) $3b-a-1 = 0$,

(iii) $(2b-1)(2b-a)-b^2 = 0$, and

(iv) $b(b-1)(b-a) = 0$.

Any attempt to solve equations (ii) - (iv) leads to an inconsistency. We therefore conclude that $T_2 \neq 0$.

Next, we assume $S_2 = 0$, whereupon we again obtain the above set of inconsistent equations, so that we conclude $S_2 \neq 0$. We have now deduced that $S_2T_2 \neq 0$, and the restriction given by equation (14) is satisfied.

We recall that h is a positive integer. Then $h + 2 - m > 2 - m$, and from identity (277) it is clear that $h + 2 - m = 4$, or $h - m = 2$. From this information we deduce the following cases to be investigated:

Case I: $h = 1$; $m = -1$

Case II: $h = 2$; $m = 0$

Case III: $h = 3$; $m = 1$.

The above cases form an exhaustive set, which may be demonstrated as follows:

Let $h \geq 4$, then $m \geq 2$. But if $m \geq 2$, we have from identity (277) $S_2 = 0$, which is contrary to previous considerations. Thus $h \leq 3$. Also h is a positive integer by Scheffé's criteria, hence $h \geq 1$. We thus conclude that the above set of cases to be investigated is exhaustive.

We will now study each case in turn.

Case I: $h = 1$; $m = -1$

Setting $j = 2$, $h = 1$, and $m = -1$ in equation (104), then equating the result to the right member of equation (273) gives us the following identity:

$$(278)\quad p_2(w) = S_2w^3 - T_2w^4 \equiv w^4 + w^3(3b-a-1) + w^2[(2b-1)(2b-a)-b^2] + w(b)(b-1)(b-a).$$

Equating coefficients of like powers of w in both members of the above identity gives us the following set of equations:

(i) $T_2 = -1$,

(ii) $S_2 = 3b-a-1$,

(iii) $(2b-1)(2b-a)-b^2 = 0$, and

(iv) $b(b-1)(b-a) = 0$.

From equations (ii) - (iv) we observe the following possibilities:

$b = 0$; $a = 0$; $S_2 = -1$,

$b = 1$; $a = 1$; $S_2 = 1$,

$b = a$; $a = 0$; $S_2 = -1$,

$b = a$; $a = 1$; $S_2 = 1$.

The above possibilities are easily reduced to the following:

$b = a = 0$; $S_2 = -1$,

$b = a = 1$; $S_2 = 1$.

We will choose $b = a = 0$, so that $S_2 = -1$, and equation (278) now becomes:

$$(279)\quad p_2(w) = w^4 - w^3.$$

Again in equation (104) we let j = 1, h = 1, and m = -1, then equate the result to the right member of equation (274) to get the following identity:

$$(280)\quad p_1(w) = S_1w^2 - T_1w^3 \equiv A_1w^3 + w^2(3A_1b+A_2) + w(3A_1b^2+2A_2b+A_3) + (A_1b^3+A_2b^2+A_3b+A_4).$$

Again we equate the coefficients of like powers of w in both members of the above identity to obtain the following set of equations:

$$(i)\quad T_1 = -A_1,$$
$$(ii)\quad S_1 = 3A_1b+A_2,$$
$$(iii)\quad 3A_1b^2+2A_2b+A_3 = 0, \text{ and}$$
$$(iv)\quad A_1b^3+A_2b^2+A_3b+A_4 = 0.$$

Since we have chosen a = b = 0, then the above equations, in conjunction with equations (2) - (5), give us immediately:

$$T_1 = -(1+\alpha+\beta),$$
$$S_1 = \delta-(1+\alpha+\beta).$$

Substituting these equations into equation (280) yields:

$$(281)\quad p_1(w) = (1+\alpha+\beta)w^3 - (1+\alpha+\beta-\delta)w^2.$$

Finally, to compute $p_0(w)$, we set j = 0, h = 1, and m = -1 in equation (104), then equate the result to the right member of equation (275) thus obtaining the following identity:

$$(282)\quad p_0(w) = S_0w - T_0w^2 \equiv B_1w^2 + w(2B_1b+B_2) + (B_1b^2+B_2b+B_3).$$

Equating coefficients of like powers of w in the above identity gives us the following equations:

(i) $T_0 = -B_1$,

(ii) $S_0 = 2B_1b+B_2$, and

(iii) $B_1b^2+B_2b+B_3 = 0$.

We have chosen b = 0, therefore, from equations (6) - (8) we get the following values for $T_0$ and $S_0$:

$T_0 = \alpha\beta$,

$S_0 = -\alpha\beta p$.

We note from equation (iii) that $B_3 = 0$. This implies pq = 0, since $\alpha\beta \neq 0$. We choose q = 0, $p \neq 0$, hence $S_0 = -\alpha\beta p$. Substituting the above values of $S_0$ and $T_0$ into equation (282) gives us the following expression for $p_0(w)$:

(283) $p_0(w) = \alpha\beta w^2 - \alpha\beta pw$.

Substituting equations (279), (281), and (283) into equation (269) gives us the following confluent differential equation:

$$(284)\quad w^3(w-1)\frac{d^2u}{dw^2} + [(1+\alpha+\beta)w^3 - (1+\alpha+\beta-\delta)w^2]\frac{du}{dw} + \alpha\beta w(w-p)u = 0.$$

Division of both members of equation (284) by w gives us:

$$(285)\qquad w^2(w-1)\frac{d^2u}{dw^2} + [(1+\alpha+\beta)w^2 - (1+\alpha+\beta-\delta)w]\frac{du}{dw} + \alpha\beta(w-p)u = 0.$$

We will now classify equation (285) in the notation of Ince.

Let us initially write equation (285) in the form given by equation (117), that is:

$$(117)\qquad \frac{d^2u}{dw^2} + p(w)\frac{du}{dw} + q(w)u = 0.$$

Upon comparison with equation (285), we note the following correspondences:

$$(286)\qquad p(w) = \frac{(1+\alpha+\beta)w^2 - (1+\alpha+\beta-\delta)w}{w^2(w-1)}, \text{ and}$$

$$(287)\qquad q(w) = \frac{\alpha\beta(w-p)}{w^2(w-1)}.$$

We immediately observe from equations (286) and (287) that $wp(w) = 1+\alpha+\beta-\delta$, and $w^2q(w) = \alpha\beta p$ when $w = 0$, so that $w = 0$ is a regular singular point of equation (285). Again we note from equations (286) and (287) that $(w-1)p(w) = \delta$, and $(w-1)^2q(w) = 0$ when $w = 1$. Thus $w = 1$ is a regular singular point of equation (285).

In order to investigate whether or not infinity is a regular singular point of equation (285), we initially make the variable transformation $v = 1/w$, substitute into equation (285), and then investigate whether or not $v = 0$

is a regular singular point of the resulting equation. Subject to the above variable transformation, equation (117) becomes:

$$(288)\qquad \frac{d^2u}{dv^2} + [2/v - (1/v^2)p(1/v)]\frac{du}{dv} + (1/v^4)q(1/v)u = 0.$$

We now recall from Chapter I, that if both $v[2/v - (1/v^2)p(1/v)]$ and $v^2(1/v^4)q(1/v)$ are analytic at $v = 0$, then $v = 0$ is a regular singular point of equation (288), which implies that infinity is a regular singular point of equation (285).

From equation (286) we note the following:

$$p(1/v) = \frac{v(1+\alpha+\beta)}{1 - v} - \frac{v^2(1+\alpha+\beta-\delta)}{1 - v}, \text{ hence}$$

$$(289)\qquad v[2/v - (1/v^2)p(1/v)] = 2 - \frac{(1+\alpha+\beta)}{1 - v} + \frac{v(1+\alpha+\beta-\delta)}{1 - v}.$$

Also, $$q(1/v) = \frac{\alpha\beta v^2(1-pv)}{1 - v}, \text{ and thus}$$

$$(290)\qquad (1/v^2)q(1/v) = \frac{\alpha\beta(1-pv)}{1 - v}.$$

From equations (289) and (290) we see that

$v[2/v - (1/v^2)p(1/v)] = 1-\alpha-\beta$, and

$(1/v^2)q(1/v) = \alpha\beta$ when $v = 0$, so that $v = 0$ is a regular singular point of equation (288), and hence infinity is a regular singular point of equation (285).

From the above, we see that w = 0,1, and infinity are regular singular points of equation (285). We have thus examined all points at which equation (285) fails to be analytic. We will now determine whether or not the preceding regular singular points are elementary singular points.

Assume a solution of equation (285) in a neighborhood of w = 0 to be of the form:

$$(291) \quad u(w) = c_0 w^{\lambda} + c_1 w^{\lambda+1} + \cdots + c_r w^{\lambda+r} + \cdots; \; c_0 \neq 0.$$

Differentiate equation (291) with respect to w to get:

$$(292) \quad u'(w) = c_0 \lambda w^{\lambda-1} + c_1(\lambda+1) w^{\lambda} + \cdots + c_r(\lambda+r) w^{\lambda+r-1} + \cdots .$$

Differentiate equation (292) with respect to w to get:

$$(293) \quad u''(w) = c_0 \lambda(\lambda-1) w^{\lambda-2} + c_1(\lambda+1)(\lambda) w^{\lambda-1} + \cdots + c_r(\lambda+r)(\lambda+r-1) w^{\lambda+r-2} + \cdots .$$

Substitute equations (291) - (293) into equation (285) and equate the coefficient of $w^{\lambda}$ to zero to get the following indicial equation:

$$(294) \quad -c_0 \lambda(\lambda-1) - c_0 \lambda(1+\alpha+\beta-\delta) - c_0 \alpha\beta p = 0.$$

Solving equation (294) for $\lambda$, gives us:

$$\lambda_1 = (\delta-\alpha-\beta-R)/2, \text{ and}$$

$$\lambda_2 = (\delta-\alpha-\beta+R)/2, \text{ where}$$

$$R = \sqrt{(\delta-\alpha-\beta)^2 - 4\alpha\beta p}.$$

We note that $\lambda_2 - \lambda_1 = R \neq 1/2$; therefore, $w = 0$ is not an elementary singular point of equation (285).

In order to investigate whether or not $w = 1$ is an elementary singular point of equation (285), we first make the variable change $v = w-1$. Equation (285) then becomes:

$$(295) \quad v(v+1)^2 \frac{d^2u}{dv^2} + [v^2(1+\alpha+\beta) + v(1+\alpha+\beta+\delta) + \delta]\frac{du}{dv} + \alpha\beta(v+1-p)u = 0.$$

We now assume a solution of equation (295) in the form given by equation (291) with w replaced by v. We then replace w by v in equations (292) and (293), then substitute these three replaced equations into equation (295), and equate the coefficient of $v^{\lambda-1}$ to zero to get the following indicial equation:

$$(296) \quad c_o\lambda(\lambda-1) + c_o\lambda\delta = 0, \text{ hence}$$

$$\lambda = 0, 1 - \delta.$$

These are the exponents at $v = 0$. But $v = 0$ implies $w = 1$, which further implies $z = 1$, since we have chosen $a = 0$. Therefore, the above values of $\lambda$ are the exponents at $z = 1$.

Prior to investigating whether or not infinity is an elementary singular point of equation (285), we make the variable transformation $v = 1/w$. The differential equation resulting from this transformation follows immediately from equations (288) - (290), thus:

$$(297)\quad \frac{d^2u}{dv^2} + \left[\frac{2}{v} - \frac{1+\alpha+\beta}{v(1-v)} + \frac{1+\alpha+\beta-\delta}{1-v}\right]\frac{du}{dv} + \frac{\alpha\beta(1-pv)}{v^2(1-v)}u = 0.$$

Equation (297) may be written as follows:

$$(298)\quad v^2(1-v)\frac{d^2u}{dv^2} + \left[2v(1-v) - v(1+\alpha+\beta) + v^2(1+\alpha+\beta-\delta)\right]\frac{du}{dv} + \alpha\beta(1-pv)u = 0.$$

As before, we replace w by v in equations (291) - (293), substitute into equation (298), then equate the coefficient of $v^\lambda$ to zero to obtain the following indicial equation:

$$(299)\quad c_o\lambda(\lambda-1) + c_o\lambda(2-1-\alpha-\beta) + c_o\alpha\beta = 0.$$

Solving equation (299) for $\lambda$ gives us:

$$\lambda_1 = \alpha; \qquad \lambda_2 = \beta.$$

Evidently $\lambda_2 - \lambda_1 = \beta - \alpha \neq 1/2$; therefore, $v = 0$ is not an elementary singular point of equation (298), which implies that infinity is not an elementary singular point of equation (285).

Since there are no elementary singular points, three regular singular points, and no irregular singularities of equation (285), we classify it as an equation of the [0,3,0] type.

We will now investigate Case II, that is, $h = 2$, $m = 0$.

Case II: $h = 2$; $m = 0$

In equation (104), we let $j = 2$, $h = 2$, and $m = 0$, then equate the result to the right member of equation (273), thus obtaining the following identity:

$$(300)\quad p_2(w) = S_2w^2 - T_2w^4 \equiv w^4 + w^3(3b-a-1) + w^2[(2b-1)(2b-a)-b^2] + w(b)(b-1)(b-a).$$

Equating coefficients of like powers of w in both members of the above identity, gives us the following set of equations:

(i) $T_2 = -1$,

(ii) $S_2 = (2b-1)(2b-a)-b^2$,

(iii) $3b-a-1 = 0$, and

(iv) $b(b-1)(b-a) = 0$.

From equations (ii) - (iv), we have the following set of values for b, a, and $S_2$:

$b = 0$; $a = -1$; $S_2 = -1$,

$b = 1$; $a = 2$; $S_2 = -1$, and

$b = a$; $a = 1/2$; $S_2 = -1/4$.

Of the above values, we choose $b = 0$, $a = -1$, and $S_2 = -1$, so that equation (300) becomes:

$$(301)\quad p_2(w) = w^4 - w^2.$$

In order to compute $p_1(w)$, we let $j = 1$, $h = 2$, and $m = 0$ in equation (104), then equate the result to the right member of equation (274), to obtain the following identity:

$$(302)\quad p_1(w) = S_1w - T_1w^3 \equiv A_1w^3 + w^2(3A_1b+A_2) + w(3A_1b^2+2A_2b+A_3) + (A_1b^3+A_2b^2+A_3b+A_4).$$

Equating coefficients of like powers of w in both members of identity (302), gives us the following set of equations:

$$(i)\quad T_1 = -A_1,$$

$$(ii)\quad S_1 = 3A_1b^2+2A_2b+A_3,$$

$$(iii)\quad 3A_1b+A_2 = 0, \text{ and}$$

$$(iv)\quad A_1b^3+A_2b^2+A_3b+A_4 = 0.$$

Referring to equations (2) - (5), and remembering that b = 0, a = -1, the above equations become:

$$T_1 = -(1+\alpha+\beta),$$

$$S_1 = -(\gamma+\eta),$$

$$\delta = \epsilon, \text{ and}$$

$$A_4 = -ab\gamma = 0.$$

Equation (302) now becomes:

$$(303)\quad p_1(w) = (1+\alpha+\beta)w^3 - (\gamma+\eta)w.$$

We now compute $p_0(w)$, by letting j = 0, h = 2, and m = 0 in equation (104). We then equate the result to the right member of equation (275), thus obtaining the following identity:

$$(304)\quad p_0(w) = S_0 - T_0w^2 \equiv B_1w^2+w(2B_1b+B_2) + (B_1b^2+B_2b+B_3).$$

Again we equate the coefficients of like powers of w in both members of the above identity to obtain the following set of equations:

(i) $T_o = -B_1$,

(ii) $S_o = B_1b^2+B_2b+B_3$, and

(iii) $2B_1b+B_2 = 0$.

From equation (iii) we note that $B_2 = 0$, since we have previously chosen $b = 0$. This implies from equation (7) that $p = -q$, since $\alpha\beta \neq 0$. Again from equations (i) and (ii) above, and also from equations (6) - (8), we deduce the following values for $T_o$ and $S_o$:

$T_o = -\alpha\beta$, and

$S_o = -\alpha\beta p^2$.

Substitution of these values for $T_o$ and $S_o$ into equation (304), gives the following expression for $p_o(w)$:

(305) $p_o(w) = \alpha\beta(w^2-p^2)$.

Substituting equations (301), (303), and (305) into equation (269), gives us the following differential equation:

(306) $w^2(w^2-1)\frac{du}{dw} + [(1+\alpha+\beta)w^3 - (\gamma+\eta)w]\frac{du}{dw} + \alpha\beta(w^2-p^2)u = 0.$

Equation (306) is the result of applying Scheffé's criteria to equation (276), where $h = 2$, $m = 0$. We chose $b = 0$, $a = -1$, and $q = -p$. Other choices of a and b will produce other confluent differential equations. Although we will

not classify equation (306), we mention that it can be classified in a manner similar to that used to classify equation (285).

We will investigate next Case III, that is, $h = 3$, $m = 1$.

Case III: $h = 3;\ m = 1$

In equation (104), we let $j = 2$, $h = 3$, and $m = 1$. We then equate the result to the right member of equation (273) thus obtaining the following identity:

$$(307)\quad p_2(w) = S_2w - T_2w^4 \equiv w^4 + w^3(3b-a-1) + w^2[(2b-1)(2b-a)-b^2] + w(b)(b-1)(b-a).$$

Equating coefficients of like powers of w in both members of the above identity gives us the following set of equations:

(i) $T_2 = -1$,

(ii) $S_2 = b(b-1)(b-a)$,

(iii) $3b-a-1 = 0$, and

(iv) $(2b-1)(2b-a)-b^2 = 0$.

From equation (iii), we have $b = \frac{a+1}{3}$. Substitute this value of b into equation (iv) to get:

$[1 - (2a+2)/3][a - (2a+2)/3] - [(a+1)/3]^2 = 0$, or

$(1-2a)(a-2) - (a+1)^2 = 0$, which simplifies to

$a^2 - a + 1 = 0$.

The solution of this equation, in conjunction with the equation $b = (a+1)/3$, gives us the following set of equations:

$$(308)\quad \begin{cases} a_1 = -\omega^2; & b_1 = (1-\omega^2)/3, \\ a_2 = -\omega; & b_2 = (1-\omega)/3, \text{ where} \\ 1+\omega+\omega^2 = 0, & \omega = (-1+i\sqrt{3})/2, \text{ and } i^2 = -1. \end{cases}$$

Other combinations of a and b needed are readily deduced, thus:

$$(309)\quad \begin{cases} b_1a_1 = i\sqrt{3}/3; & b_2a_2 = -i\sqrt{3}/3, \\ b_1+a_1 = (3+2i\sqrt{3})/3; & b_2+a_2 = (3-2i\sqrt{3})/3, \text{ and} \\ b_1^3 = i\sqrt{3}/9; & b_2^3 = -i\sqrt{3}/9. \end{cases}$$

We will choose $a_1 = -\omega^2$, $b_1 = (1-\omega^2)/3$, and hence we can deduce a value for $S_2$ from equation (ii), thus:

$$S_2 = b_1(b_1-1)b_1-a_1)$$
$$= b_1(b_1^2-a_1b_1-b_1+a_1), \text{ or}$$
$$S_2 = b_1^3-a_1b_1^2-b_1^2+a_1b_1.$$

From equations (309), this simplifies to

$$S_2 = b_1^3-b_1^2(a_1+1) + a_1b_1$$
$$= b_1^3-3b_1^3+a_1b_1, \text{ since } a_1+1 = 3b_1. \text{ Then}$$
$$S_2 = a_1b_1 - 2b_1^3,$$
$$= (i\sqrt{3}/3) - (2i\sqrt{3}/9), \text{ or finally,}$$
$$S_2 = i\sqrt{3}/9.$$

Equation (307) now becomes:

$$(310)\quad p_2(w) = w^4 + (i\sqrt{3}/9)w.$$

To compute $p_1(w)$, we set $j = 1$, $h = 3$, and $m = 1$ in equation (104). We then equate the result to the right member of equation (274), to obtain the following identity:

$$(311)\quad p_1(w) = S_1 - T_1w^3 \equiv A_1w^3 + w^2(3A_1b+A_2) + w(3A_1b^2+2A_2b+A_3) + (A_1b^3+A_2b^2+A_3b+A_4).$$

Equating coefficients of like powers of w in both members of the above identity, we obtain the following set of equations:

$$(i)\quad T_1 = -A_1,$$
$$(ii)\quad S_1 = A_1b^3+A_2b^2+A_3b+A_4,$$
$$(iii)\quad 3A_1b+A_2 = 0, \text{ and}$$
$$(iv)\quad 3A_1b^2+2A_2b+A_3 = 0.$$

From equations (ii) - (iv), we easily deduce that $A_2b+A_3 = 0$, and hence that $S_1 = A_1b^3+A_4$. We note from equations (2) - (5) that $A_1 = 1+\alpha+\beta$, and $A_4 = -ab\gamma$. Also from equations (309) we have $a_1b_1 = i\sqrt{3}/3$, and $b_1^3 = i\sqrt{3}/9$. Thus we finally have for $T_1$ and $S_1$:

$$T_1 = -(1+\alpha+\beta), \text{ and}$$
$$S_1 = (i\sqrt{3}/9)(1+\alpha+\beta) - (i\sqrt{3}/3)\gamma.$$

Equation (311) now becomes:

$$(312)\quad p_1(w) = (1+\alpha+\beta)w^3 + (i\sqrt{3}/9)(1+\alpha+\beta) - (i\sqrt{3}/3)\gamma.$$

Finally we compute $p_0(w)$ from equation (104) by letting $j = 0$, $h = 3$, and $m = 1$. Then, as we have done previously, we equate the result to the right member of equation (275), thus obtaining the following identity:

$$(313)\quad p_o(w) = S_o w^{-1} - T_o w^2 \equiv B_1 w^2 + w(2B_1 b + B_2) + (B_1 b^2 + B_2 b + B_3).$$

Equating coefficients of like powers of w in both members of the above identity, gives us the following set of equations:

$$(i)\quad T_o = -B_1,$$
$$(ii)\quad S_o = 0,$$
$$(iii)\quad 2B_1 b + B_2 = 0, \text{ and}$$
$$(iv)\quad B_1 b^2 + B_2 b + B_3 = 0.$$

From equation (6), $B_1 = \alpha\beta$, hence $T_o = -\alpha\beta$, and equation (313) now becomes:

$$(314)\quad p_o(w) = \alpha\beta w^2.$$

Substituting equations (310), (312), and (314) into equation (269), gives us the following differential equation:

$$(315)\quad w(w^3 + i\sqrt{3}/9)\frac{d^2u}{dw^2} + [(1+\alpha+\beta)w^3 + (i\sqrt{3}/9)(1+\alpha+\beta) - (i\sqrt{3}/3)\gamma]\frac{du}{dw} + \alpha\beta w^2 u = 0.$$

Prior to solving equation (315), we will first show the following:

$$(316)\quad 1 + \alpha + \beta - 3\gamma = \eta.$$

A rather important by-product of the computation of $S_1$ from identity (311) was the following equation:

$$(316)\quad A_2 b + A_3 = 0.$$

Substituting the values of $A_2$ and $A_3$ from equations (3) and (4) into equation (316) yields:

(317) $b[(1+\alpha+\beta)(1+a+b) - a\epsilon - b\eta - \delta]$

$= [ab(\gamma+\delta) + a(\gamma+\eta) + b(\gamma+\epsilon)].$

But we recall from equation (307 iii) that $b = (a+1)/3$, hence substituting into equation (317) and rearranging slightly gives us:

$(1+\alpha+\beta)[1+a+(a+1)/3] - a\epsilon - \eta[(a+1)/3] - \delta$

$= a(\gamma+\delta) + (a/b)(\gamma+\eta) + (\gamma+\epsilon).$

Simplifying the above we get:

$(1+\alpha+\beta)[(4+4a)/3] - \epsilon(a+1) - \eta[(a+1)/3] - \gamma(a+1)+\delta(a+1)$

$= (a/b)(\gamma+\eta).$

Again we recall that $b = (a+1)/3$, so that the last written equation now becomes, upon slight rearrangement:

$(3b)(1+\alpha+\beta-\epsilon-\delta-\gamma-\eta) = -2b\eta + (a/b)(\gamma+\eta) - b(1+\alpha+\beta).$

Now from equation (12) in Chapter V, we saw that $(1+\alpha+\beta-\epsilon-\delta-\gamma-\eta) = 0$; therefore, the above equation becomes:

$b(1+\alpha+\beta) + 2b\eta = (a/b)(\gamma+\eta).$

Multiply both members of this equation by $b^2$, and rearrange slightly to obtain:

$b^3(1+\alpha+\beta) - ab\gamma = \eta(ab-2b^3).$

We now observe from equations (309), that $b_1^3 = i\sqrt{3}/9$, $a_1b_1 = i\sqrt{3}/3$, hence substitute above to get the following equation:

$$(i\sqrt{3}/9)(1+\alpha+\beta) - (i\sqrt{3}/3)\gamma = \eta[(i\sqrt{3}/3) - 2(i\sqrt{3}/9)], \text{ or}$$

$$(318) \quad (i\sqrt{3}/9)(1+\alpha+\beta) - (i\sqrt{3}/3)\gamma = \eta(i\sqrt{3}/9).$$

Division of both members of equation (318) by $(i\sqrt{3}/9)$ gives us equation (316).

Equation (315) can now be written:

$$(319) \quad w(w^3+i\sqrt{3}/9)\frac{d^2u}{dw^2} + [(1+\alpha+\beta)w^3 + (i\sqrt{3}/9)\eta]\frac{du}{dw} + \alpha\beta w^2 u = 0.$$

Equation (319) is the equation which we will now solve. It was derived on the basis of our choice of $a_1 = -\omega^2$, and $b_1 = (1-\omega^2)/3$. Other choices for a and b will result in other non-confluent differential equations. Further, equation (319) is classified in the notation of Ince as an equation of the [0,5,0] class, since the regular singular points are distinct non-elementary singular points, also it contains no irregular singular points.

We now assume a solution of equation (319) in the form given by equation (18). Differentiate equation (18) with respect to w to get equation (19), then differentiate equation (19) with respect to w to get equation (20). We then substitute equations (18) - (20) into equation (319). The resulting equation follows:

$$(320)\quad w(w^3+i\sqrt{3}/9)[c_0\lambda(\lambda-1)w^{\lambda-2} + c_3(\lambda+3)(\lambda+2)w^{\lambda+1} + \cdots$$
$$+ c_{3r}(\lambda+3r)(\lambda+3r-1)w^{\lambda+3r-2}$$
$$+ c_{3r+3}(\lambda+3r+3)(\lambda+3r+2)w^{\lambda+3r+1} + \cdots]$$
$$+ [(1+\alpha+\beta)w^3 + (i\sqrt{3}/9)\eta][c_0\lambda w^{\lambda-1} + c_3(\lambda+3)w^{\lambda+2} + \cdots$$
$$+ c_{3r}(\lambda+3r)w^{\lambda+3r-1}$$
$$+ c_{3r+3}(\lambda+3r+3)w^{\lambda+3r+2} + \cdots]$$
$$+ (\alpha\beta w^2)[c_0w^{\lambda} + c_3w^{\lambda+3} + \cdots + c_{3r}w^{\lambda+3r}$$
$$+ c_{3r+3}w^{\lambda+3r+3} + \cdots] = 0;\ c_0 \neq 0.$$

The indicial equation is found by equating the coefficient of $w^{\lambda-1}$ to zero, thus:

$$(321)\quad c_0\lambda(\lambda-1)(i\sqrt{3}/9) + c_0\lambda\eta(i\sqrt{3}/9) = 0.$$

Solving equation (321) for $\lambda$, yields the following expression:

$$(322)\quad \lambda_1 = 0; \qquad \lambda_2 = 1 - \eta.$$

These are the exponents at $w = 0$, that is, at $z = b$. This verifies our choice of exponents at $z = b$.

To obtain the desired two term recurrence equation, we equate the coefficient of $w^{\lambda+3r+2}$ to zero, thus:

$$(323)\quad c_{3r}(\lambda+3r)(\lambda+3r-1) + (i\sqrt{3}/9)c_{3r+3}(\lambda+3r+3)(\lambda+3r+2)$$
$$+ (1+\alpha+\beta)(c_{3r})(\lambda+3r) + (i\sqrt{3}/9)(\eta)(c_{3r+3})(\lambda+3r+3)$$
$$+ \alpha\beta c_{3r} = 0.$$

Solving equation (323) for $c_{3r+3}$ yields:

$$(324)\quad c_{3r+3} = \frac{-[(\lambda+3r)(\lambda+3r-1) + (\lambda+3r)(1+\alpha+\beta) + \alpha\beta]}{(i\sqrt{3}/9)[(\lambda+3r+3)(\lambda+3r+2) + \eta(\lambda+3r+3)]}c_{3r}.$$

Upon slight simplification, equation (324) can be written as follows:

$$(325)\quad c_{3r+3} = \frac{(3i\sqrt{3})(\lambda+3r+\alpha)(\lambda+3r+\beta)}{(\lambda+3r+3)(\lambda+3r+2+\eta)}c_{3r}.$$

Equation (325) is the desired form of the two term recurrence equation.

In equation (325), we initially set r = 0, then we have:

$$(326)\quad c_3 = \frac{(3i\sqrt{3})(\lambda+\alpha)(\lambda+\beta)}{(\lambda+3)(\lambda+3+\eta)}c_0.$$

Next, set r = 1 in equation (325) to get:

$$c_6 = \frac{(3i\sqrt{3})(\lambda+3+\alpha)(\lambda+3+\beta)}{(\lambda+3+3)(\lambda+3+2+\eta)}c_3.$$

Substituting equation (326) into the above equation gives us $c_6$ in terms of $c_o$, thus:

$$(327)\quad c_6 = \frac{(3i\sqrt{3})^2(\lambda+\alpha)(\lambda+\alpha+3)(\lambda+\beta)(\lambda+\beta+3)}{(\lambda+3)(\lambda+3+3)(\lambda+2+\eta)(\lambda+2+\eta+3)}c_o.$$

Finally, letting r = n - 1 in equation (325) gives the following equation:

$$(328)\quad c_{3n} = \frac{(3i\sqrt{3})^n(\lambda+\alpha)\cdots(\lambda+\alpha+3n-3)(\lambda+\beta)\cdots(\lambda+\beta+3n-3)}{(\lambda+3)(\lambda+3+3)\cdots(\lambda+3+3n-3)(\lambda+2+\eta)\cdots(\lambda+2+\eta+3n-3)}c_o.$$

Dividing numerator and denominator of equation (328) by $3^{2n}$, gives us the following version:

$$(329)\quad c_{3n} = \frac{(3i\sqrt{3})^n(\frac{\lambda+\alpha}{3})\cdots(\frac{\lambda+\alpha}{3}+n-1)(\frac{\lambda+\beta}{3})\cdots(\frac{\lambda+\beta}{3}+n-1)}{(\frac{\lambda+3}{3})(\frac{\lambda+3}{3}+1)\cdots(\frac{\lambda+3}{3}+n-1)(\frac{\lambda+2+\eta}{3})\cdots(\frac{\lambda+2+\eta}{3}+n-1)}c_o.$$

We now write a solution of equation (319) in the form:

$$(330)\quad \bar{u}(w) = c_o w^{\lambda}\left[1 + \sum_{n=1}^{\infty}(c_{3n}/c_o)w^{3n}\right].$$

Now for $\lambda = 0$, we can substitute equation (329) into (330) thus obtaining the following solution to equation (319):

$$(331)\quad \bar{u}_1(w) = \bar{K}_1\left[1 + \sum_{n=1}^{\infty}\frac{(3i\sqrt{3})^n(\frac{\alpha}{3})\cdots(\frac{\alpha}{3}+n-1)(\frac{\beta}{3})\cdots(\frac{\beta}{3}+n-1)}{n!(\frac{2+\eta}{3})(\frac{2+\eta}{3}+1)\cdots(\frac{2+\eta}{3}+n-1)}w^{3n}\right].$$

Again we recall the expression for the hypergeometric function as given by equation (84), and we set up the following correspondences:

$$a = \alpha/3; \qquad c = (2+\eta)/3,$$

$$b = \beta/3; \qquad z = 3i\sqrt{3}w^3.$$

We now write equation (331) in terms of hypergeometric functions, thus:

$$(332)\quad \bar{u}_1(w) = \bar{K}_1 F[\alpha/3,\ \beta/3;\ (2+\eta)/3;\ 3i\sqrt{3}w^3].$$

Finally, we recall that $w = z - b$, hence we let

$$u_1(z) = \bar{u}_1(z-b), \text{ so that equation (332) becomes:}$$

$$(333)\quad u_1(z) = \bar{K}_1 F[\alpha/3,\ \beta/3;\ (2+\eta)/3;\ 3i\sqrt{3}(z-b)^3].$$

We now let $\lambda = 1 - \eta$, so that equation (329) becomes:

$$(334)\quad c_{3n} = \frac{(3i\sqrt{3})^n(\frac{1-\eta+\alpha}{3})\cdots(\frac{1-\eta+\alpha}{3}+n-1)(\frac{1-\eta+\beta}{3})\cdots(\frac{1-\eta+\beta}{3}+n-1)}{n!(\frac{4-\eta}{3})(\frac{4-\eta}{3}+1)\cdots(\frac{4-\eta}{3}+n-1)}c_o.$$

Substituting equation (334) into equation (331) with $\lambda = 1 - \eta$ gives us the following equation:

$$(335)\quad \bar{u}_2(w) = \bar{K}_2 w^{1-\eta}(1+S), \text{ where}$$

$$S = \sum_{n=1}^{\infty} \frac{(3i\sqrt{3})^n(\frac{1-\eta+\alpha}{3})\cdots(\frac{1-\eta+\alpha}{3}+n-1)(\frac{1-\eta+\beta}{3})\cdots(\frac{1-\eta+\beta}{3}+n-1)}{n!(\frac{4-\eta}{3})(\frac{4-\eta}{3}+1)\cdots(\frac{4-\eta}{3}+n-1)}w^{3n}.$$

Again we see that equation (335) is expressible in terms of the hypergeometric function, if we first establish the following correspondences:

$$a = (1-\eta+\alpha)/3; \qquad c = (4-\eta)/3,$$

$$b = (1-\eta+\beta)/3; \qquad z = 3i\sqrt{3}w^3.$$

Subject to these correspondences, equation (335) becomes:

$$(336)\quad \bar{u}_2(w) = \bar{K}_2 w^{1-\eta} F[\tfrac{1-\eta+\alpha}{3}, \tfrac{1-\eta+\beta}{3}; \tfrac{4-\eta}{3}; 3i\sqrt{3}w^3].$$

Again, $\bar{u}_2(w) = \bar{u}_2(z-b) = u_2(z)$, hence replacing w by (z-b), gives us equation (336) in its desired form, thus:

$$(337)\quad u_2(z) = \bar{K}_2(z-b)^{1-\eta} F[\tfrac{1-\eta+\alpha}{3}, \tfrac{1-\eta+\beta}{3}; \tfrac{4-\eta}{3}; 3i\sqrt{3}(z-b)^3].$$

A general solution of equation (319) is a linear combination of equations (333) and (336), thus:

$$(338)\quad u(z) = K_1 F[\alpha/3,\ \beta/3;\ (2+\eta)/3;\ 3i\sqrt{3}(z-b)^3] + K_2(z-b)^{1-\eta}F\left[\frac{1-\eta+\alpha}{3},\ \frac{1-\eta+\beta}{3};\ \frac{4-\eta}{3};\ 3i\sqrt{3}(z-b)^3\right].$$

In equation (338) we note that $\eta \neq 1 \pm 3n$; $(n = 1,2,\ldots)$.

It can be shown by the ratio test that the solution of equation (319) as given by equation (338) is absolutely convergent in the open disk

$$|z-b| < \frac{1}{|3i\sqrt{3}|^{1/3}}, \text{ that is, } |z-b| < \frac{\sqrt{3}}{3}.$$

We chose $b_1 = (1-\omega^2)/3$ and $a_1 = -\omega^2$, so that the solution denoted by equation (338) is absolutely convergent, provided $|z - (1-\omega^2)/3| < \frac{\sqrt{3}}{3}$.

We will now apply Scheffé's criteria to equation (1), and investigate a solution of the resulting equation in a neighborhood of infinity.

## Solution in a Neighborhood of the Singular Point at Infinity

In order to apply Scheffé's criteria to equation (1) and investigate a solution of the result in a neighborhood of infinity, we initially make the following variable transformation:

(339) $z = 1/w$.

We will then substitute equation (339) into equation (1), along with appropriate derivatives, apply Scheffé's criteria to the result, and then investigate a solution of the resulting differential equation in a neighborhood of $w = 0$.

From equation (339), we deduce the following useful equations:

(340) $\frac{du}{dw} = \frac{du}{dz}\cdot\frac{dz}{dw}$, where $\frac{dz}{dw} = -1/w^2$.

We also have

(341) $\frac{d^2u}{dw^2} = \frac{du}{dz}\cdot\frac{d^2z}{dw^2} + \frac{d^2u}{dz^2}\cdot\left[\frac{dz}{dw}\right]^2$, where

$$\frac{d^2z}{dw^2} = 2/w^3.$$

Equations (340) and (341) now become:

(342) $\frac{du}{dw} = -(1/w^2)\frac{du}{dz}$, and

(343) $\frac{d^2u}{dw^2} = (2/w^3)\frac{du}{dz} + (1/w^4)\frac{d^2u}{dz^2}$.

Solving equations (342) and (343) for $\frac{du}{dz}$ and $\frac{d^2u}{dz^2}$, respectively, gives us the following equations:

(344) $\frac{du}{dz} = -w^2 \frac{du}{dw}$, and

(345) $\frac{d^2u}{dz^2} = w^4 \frac{d^2u}{dw^2} + 2w^3 \frac{du}{dw}$.

We now substitute equations (339), (344), and (345) into equation (1) to get the following equation:

(346) $$(\tfrac{1}{w})(\tfrac{1}{w}-1)(\tfrac{1}{w}-a)(\tfrac{1}{w}-b)\left[w^4 \frac{d^2u}{dw^2} + 2w^3 \frac{du}{dw}\right] + \left[\frac{A_1}{w^3} + \frac{A_2}{w^2} + \frac{A_3}{w} + A_4\right]\left[-w^2 \frac{du}{dw}\right] + \left[\frac{B_1}{w^2} + \frac{B_2}{w} + B_3\right]u = 0.$$

Multiplying both members of equation (346) by $w^2$, gives us the following equation:

(347) $$w^2(1-w)(1-aw)(1-bw)\frac{d^2u}{dw^2} + [2w(1-w)(1-aw)(1-bw) - A_1w - A_2w^2 - A_3w^3 - A_4w^4]\frac{du}{dw} + (B_1+B_2w+B_3w^2)u = 0.$$

Equation (347) may be reduced to the following form after some algebraic manipulation:

$$(348)\quad [abw^5 - w^4(a+b+ab) + w^3(a+b+1) - w^2]\frac{d^2u}{dw^2}$$
$$+ [w^4(2ab+A_4) - w^3(2a+2b+2ab-A_3)$$
$$+ w^2(2a+2b+2+A_2) - w(2-A_1)]\frac{du}{dw}$$
$$- [B_3w^2+B_2w+B_1]u = 0.$$

We now write equation (348) as follows:

$$(349)\quad p_2(w)\frac{d^2u}{dw^2} + p_1(w)\frac{du}{dw} + p_o(w)u = 0,$$

where we note that

$$(350)\quad p_2(w) = abw^5 - w^4(a+b+ab) + w^3(a+b+1) - w^2,$$

$$(351)\quad p_1(w) = w^4(2ab+A_4) - w^3(2a+2b+2ab-A_3)$$
$$+ w^2(2a+2b+2+A_2) - w(2-A_1), \text{ and}$$

$$(352)\quad p_o(w) = -[B_3w^2+B_2w+B_1].$$

We will now use Scheffé's criteria in the form of equation (104). Initially we let j = 2, then equate the result to the right member of equation (350) to obtain the following identity:

$$(353)\quad p_2(w) = S_2w^{2-m} - T_2w^{h+2-m} \equiv abw^5 - w^4(a+b+ab)$$
$$+ w^3(a+b+1) - w^2.$$

Prior to completing the application of Scheffé's criteria to equation (348), we will justify the restriction

given by equation (14). Thus, in equation (353) we assume $T_2 = 0$. Then $2 - m = 5$, and we are led to the following set of equations from identity (353):

(i) $S_2 = ab$,

(ii) $a+b+ab = 0$,

(iii) $a+b+1 = 0$, and

(iv) $-1 = 0$.

This obvious inconsistency leads us to the conclusion $T_2 \neq 0$.

Again, in identity (353), we assume $S_2 = 0$. Then $h + 2 - m = 5$, and upon equating the coefficients of like powers of w in both members, we obtain the following set of equations:

(i) $T_2 = -ab$,

(ii) $a+b+ab = 0$,

(iii) $a+b+1 = 0$, and

(iv) $-1 = 0$.

From these equations we conclude that $S_2 \neq 0$, and hence $S_2T_2 \neq 0$ so that the restriction given by equation (14) is satisfied.

From equation (353) we note that $h + 2 - m > 2 - m$ since h is a positive integer. Also $h + 2 - m = 5$, otherwise $ab = 0$ which implies $a = 0$, or $b = 0$, or $a = b = 0$, and immediately we have a differential equation with

confluent singular points. The equation $h - m = 3$ leads us to an investigation of the following possible values of h and m:

Case I: $h = 1$; $m = -2$,

Case II: $h = 2$; $m = -1$, and

Case III: $h = 3$; $m = 0$.

We will now demonstrate exhaustiveness of the above set of cases. Obviously $m \geq -2$, otherwise h is an integer less than unity, which is contrary to Scheffé's criteria. We recall from Chapter VII that m is an integer. Also $m \leq 0$, otherwise we have from equation (353) that $S_2 = 0$, which is contrary to previous considerations. The above set of cases is thus exhaustive.

We can immediately discard Cases I and II from consideration as follows: Suppose we let $h = 1$ and $m = -2$. Then identity (353) becomes:

$$S_2 w^4 - T_2 w^5 \equiv abw^5 - w^4(a+b+ab) + w^3(a+b+1) - w^2.$$

Upon equating coefficients of like powers of w in both members of the above identity, we are led to the conclusion $1 = 0$ when we equate coefficients of $w^2$. Therefore, we discard Case I. In a similar fashion we discard Case II. We now investigate Case III.

Case III: $h = 3$; $m = 0$

Let $j = 2$, $h = 3$ and $m = 0$ in equation (104), then equate the result to the right member of equation (350) to get the following identity:

(354) $p_2(w) = S_2w^2 - T_2w^5 \equiv abw^5 - w^4(a+b+ab) + w^3(a+b+1) - w^2$.

Equating coefficients of like powers of w in both members of the above identity yields the following set of equations:

(i) $T_2 = -ab$,

(ii) $S_2 = -1$,

(iii) $a+b+ab = 0$, and

(iv) $a+b+1 = 0$.

Solving equations (iii) and (iv) simultaneously gives us the following equation:

$b^2+b+1 = 0$, so that

$b_1 = \omega$; $a_1 = \omega^2$,

$b_2 = \omega^2$; $a_2 = \omega$, where $\omega = \frac{-1+i\sqrt{3}}{2}$, and $i^2 = -1$.

We will consider the following values of a and b:

$a_1 = \omega^2$ and $b_1 = \omega$, also $a_1b_1 = 1$. Equation (354) now becomes:

(355) $p_2(w) = w^5 - w^2$.

Again in equation (104) we let $j = 1$, $h = 3$, and $m = 0$ to get:

(356) $p_1(w) = S_1w^1 - T_1w^4 \equiv w^4(2ab+A_4) - w^3(2a+2b+2ab-A_3) + w^2(2a+2b+2+A_2) - w(2-A_1)$.

Equating coefficients of like powers of w, we get the following set of equations:

(i) $T_1 = -(2ab+A_4)$,

(ii) $S_1 = A_1-2$,

(iii) $A_3-(2a+2b+2ab) = 0$, and

(iv) $2a+2b+2+A_2 = 0$.

We recall that $a_1b_1 = 1$, so that $T_1 = -(2+A_4)$. In conjunction with equations (2) - (5), we have the following values for $T_1$ and $S_1$:

$T_1 = -(2-\gamma)$, and

$S_1 = \alpha+\beta-1$.

Equation (356) now becomes:

(357) $p_1(w) = (2-\gamma)w^4 + (\alpha+\beta-1)w$.

Finally, to compute $p_0(w)$, we let $j = 0$, $h = 3$, and $m = 0$ in equation (104) and equate the result to the right member of equation (352), thus obtaining:

(358) $p_0(w) = S_0-T_0w^3 \equiv -B_3w^2-B_2w-B_1$.

Equating coefficients of like powers of w in both members of the above identity gives us the following set of equations:

(i) $T_0 = 0$,

(ii) $S_0 = -B_1$, and

(iii) $B_2 = 0$.

In conjunction with equations (6) - (8), we note that $S_o = -\alpha\beta$. Equation (358) now becomes:

(359) $p_o(w) = -\alpha\beta$.

Substituting equations (355), (357), and (359) into equation (349) gives us the following differential equation:

$$(360)\quad w^2(w^3-1)\frac{d^2u}{dw^2} + [(2-\gamma)w^4 + (\alpha+\beta-1)w]\frac{du}{dw} - \alpha\beta u = 0.$$

Equation (360) was derived by letting $a_1 = \omega^2$, $b_1 = \omega$, and from equations (7) and (358 iii), $p = -q$. The other values of a and b will produce another differential equation similar to equation (360). Since the singular points of equation (360) are distinct, regular, and non-elementary, we classify it in the notation of Ince as an equation of the [0,5,0] class.

We will now solve equation (360).

We assume a solution of equation (360) to be of the form given by equation (18), and we expand in a neighborhood of w = 0. Differentiating equation (18) we get equation (19), and differentiating equation (19) gives us equation (20). Substituting equations (18) - (20) into equation (360) yields the following equation:

$$(361)\quad w^2(w^3-1)[c_0\lambda(\lambda-1)w^{\lambda-2} + c_3(\lambda+3)(\lambda+2)w^{\lambda+1} + \cdots$$
$$+ c_{3r}(\lambda+3r)(\lambda+3r-1)w^{\lambda+3r-2}$$
$$+ c_{3r+3}(\lambda+3r+3)(\lambda+3r+2)w^{\lambda+3r+1} + \cdots]$$
$$+ [(2-\gamma)w^4+(\alpha+\beta-1)w][c_0\lambda w^{\lambda-1}+c_3(\lambda+3)w^{\lambda+2} + \cdots$$
$$+ c_{3r}(\lambda+3r)w^{\lambda+3r-1}$$
$$+ c_{3r+3}(\lambda+3r+3)w^{\lambda+3r+2} + \cdots]$$
$$-\alpha\beta[c_0w^{\lambda}+c_3w^{\lambda+3} + \cdots + c_{3r}w^{\lambda+3r}$$
$$+ c_{3r+3}w^{\lambda+3r+3} + \cdots] = 0;\ c_0 \neq 0.$$

Equating the coefficient of $w^{\lambda}$ to zero gives us the following indicial equation:

$$(362)\quad -c_0\lambda(\lambda-1) + (\alpha+\beta-1)(c_0\lambda) - \alpha\beta c_0 = 0.$$

Solving equation (362) for $\lambda$ gives us:

$$(363)\quad \lambda_1 = \alpha, \text{ and } \lambda_2 = \beta.$$

This is a verification of our choice of exponents at $w = 0$.

To obtain the two term recurrence equation, we equate the coefficient of $w^{\lambda+3r+3}$ to zero in equation (361), thus:

$$(364)\quad c_{3r}(\lambda+3r)(\lambda+3r-1) - c_{3r+3}(\lambda+3r+3)(\lambda+3r+2)$$
$$+ (2-\gamma)(c_{3r})(\lambda+3r) + (\alpha+\beta-1)(c_{3r+3})(\lambda+3r+3)$$
$$-\alpha\beta c_{3r+3} = 0.$$

Solving equation (364) for $c_{3r+3}$ gives us:

$$c_{3r+3} = \frac{[(\lambda+3r)(\lambda+3r-1) + (2-\gamma)(\lambda+3r)]}{(\lambda+3r+3)(\lambda+3r+2) - (\lambda+3r+3)(\alpha+\beta-1)+\alpha\beta}\, c_{3r}, \text{ or}$$

$$(365)\quad c_{3r+3} = \frac{(\lambda+3r)(\lambda+3r+1-\gamma)}{(\lambda+3r+3-\alpha)(\lambda+3r+3-\beta)}\, c_{3r}.$$

Set r = 0 in equation (365) to get:

$$(366)\quad c_3 = \frac{(\lambda)(\lambda+1-\gamma)}{(\lambda+3-\alpha)(\lambda+3-\beta)}\, c_0.$$

Set r = 1 in equation (365) to get:

$$c_6 = \frac{(\lambda+3)(\lambda+1-\gamma+3)}{(\lambda+3-\alpha+3)(\lambda+3-\beta+3)}\, c_3.$$

Substitute equation (366) into the above equation to get:

$$(367)\quad c_6 = \frac{(\lambda)(\lambda+3)(\lambda+1-\gamma)(\lambda+1-\gamma+3)}{(\lambda+3-\alpha)(\lambda+3-\alpha+3)(\lambda+3-\beta)(\lambda+3-\beta+3)}\, c_0.$$

To generalize, we set r = n - 1, thus obtaining from equation (365):

$$(368)\quad c_{3n} = \frac{(\lambda)(\lambda+3)\cdots(\lambda+3n-3)(\lambda+1-\gamma)\cdots(\lambda+1-\gamma+3n-3)}{(\lambda+3-\alpha)\cdots(\lambda+3-\alpha+3n-3)(\lambda+3-\beta)\cdots(\lambda+3-\beta+3n-3)}\, c_0.$$

Dividing numerator and denominator by $3^{2n}$ yields:

$$(369)\quad c_{3n} = \frac{(\frac{\lambda}{3})(\frac{\lambda}{3}+1)\cdots(\frac{\lambda}{3}+n-1)(\frac{\lambda+1-\gamma}{3})(\frac{\lambda+1-\gamma}{3}+1)\cdots(\frac{\lambda+1-\gamma}{3}+n-1)}{(\frac{\lambda+3-\alpha}{3})\cdots(\frac{\lambda+3-\alpha}{3}+n-1)(\frac{\lambda+3-\beta}{3})\cdots(\frac{\lambda+3-\beta}{3}+n-1)} c_0.$$

We now write a solution of equation (360) as follows:

$$(370) \qquad \bar{u}(w) = c_o w^{\lambda}[1 + \sum_{n=1}^{\infty} (c_{3n}/c_o) w^{3n}].$$

Now from equations (363), we set $\lambda = \alpha$ in equations (369) and (370) thus obtaining:

$$(371) \qquad \bar{u}_1(w) = \bar{K}_1 w^{\alpha}(1 + S), \text{ where}$$

$$S = \sum_{n=1}^{\infty} \frac{(\frac{\alpha}{3})(\frac{\alpha}{3}+1)\cdots(\frac{\alpha}{3}+n-1)(\frac{\alpha+1-\gamma}{3})\cdots(\frac{\alpha+1-\gamma}{3}+n-1)}{n!(\frac{\alpha-\beta+3}{3})(\frac{\alpha-\beta+3}{3}+1)\cdots(\frac{\alpha-\beta+3}{3}+n-1)} w^{3n}.$$

Equation (371) can be expressed in terms of the hypergeometric function if, upon comparison with equation (84), we initially make the following correspondences:

$$a = \alpha/3; \qquad c = (\alpha-\beta+3)/3, \text{ and}$$

$$b = (\alpha-\gamma+1)/3; \qquad z = w^3.$$

Subject to these correspondences, equation (371) becomes:

$$(372) \qquad \bar{u}_1(w) = \bar{K}_1 w^{\alpha} F[\tfrac{\alpha}{3}, \tfrac{\alpha-\gamma+1}{3}; \tfrac{\alpha-\beta+3}{3}; w^3].$$

We now recall from equation (339) that $w = 1/z$, so that equation (372) becomes:

$$\bar{u}_1(w) = \bar{u}_1(1/z) = u_1(z), \text{ or}$$

$$(373) \qquad u_1(z) = \bar{K}_1 z^{-\alpha} F[\tfrac{\alpha}{3}, \tfrac{\alpha-\gamma+1}{3}; \tfrac{\alpha-\beta+3}{3}; z^{-3}].$$

Again from equations (363) we let $\lambda = \beta$, hence equation (369) becomes:

$$(374)\quad c_{3n} = \frac{(\frac{\beta}{3})(\frac{\beta}{3}+1)\cdots(\frac{\beta}{3}+n-1)(\frac{\beta-\gamma+1}{3})\cdots(\frac{\beta-\gamma+1}{3}+n-1)}{n!(\frac{\beta-\alpha+3}{3})(\frac{\beta-\alpha+3}{3}+1)\cdots(\frac{\beta-\alpha+3}{3}+n-1)}\, c_o.$$

Correspondingly, equation (371) becomes:

(375) $\bar{u}_2(w) = \bar{K}_2 w^{\beta}(1 + S)$, where

$$S = \sum_{n=1}^{\infty} \frac{(\frac{\beta}{3})\cdots(\frac{\beta}{3}+n-1)(\frac{\beta-\gamma+1}{3})\cdots(\frac{\beta-\gamma+1}{3}+n-1)}{n!(\frac{\beta-\alpha+3}{3})(\frac{\beta-\alpha+3}{3}+1)\cdots(\frac{\beta-\alpha+3}{3}+n-1)}\, w^{3n}.$$

Again, we compare equation (375) with equation (84), and establish the following correspondences:

$a = \beta/3$; $\quad c = (\beta-\alpha+3)/3$, and

$b = (\beta-\gamma+1)/3$; $\quad z = w^3$.

Subject to these correspondences, equation (375) can be written in terms of the hypergeometric function, thus:

$$(376)\quad \bar{u}_2(w) = \bar{K}_2 w^{\beta} F[\tfrac{\beta}{3},\ \tfrac{\beta-\gamma+1}{3};\ \tfrac{\beta-\alpha+3}{3};\ w^3].$$

From equation (339) we have $w = 1/z$, so that substitution into equation (376) yields:

$\bar{u}_2(w) = \bar{u}_2(1/z) = u_2(z)$, or

$$(377)\quad u_2(z) = \bar{K}_2 z^{-\beta} F[\tfrac{\beta}{3},\ \tfrac{\beta-\gamma+1}{3};\ \tfrac{\beta-\alpha+3}{3};\ z^{-3}].$$

A general solution of equation (360) is a linear combination of equations (373) and (377), thus we have:

$$(378)\quad u(z) = K_1 z^{-\alpha} F\left[\frac{\alpha}{3},\ \frac{\alpha-\gamma+1}{3};\ \frac{\alpha-\beta+3}{3};\ z^{-3}\right]$$

$$+ K_2 z^{-\beta} F\left[\frac{\beta}{3},\ \frac{\beta-\gamma+1}{3};\ \frac{\beta-\alpha+3}{3};\ z^{-3}\right].$$

In equation (378) we make the restriction $\alpha \neq \beta \pm 3n$, $(n = 1,2,\ldots)$. Further, we observe from the ratio test that equation (378) is absolutely convergent provided $|z| > 1$.

In order to extend the usefulness of Scheffé's criteria, we will now apply it to equation (1) and investigate a general solution of the resulting equation in a neighborhood of an ordinary point $z = d$, where $d \neq 0$, 1, a, b, or infinity.

## Solution in a Neighborhood of z = d, where d ≠ 0, 1, a, b, or ∞

In order to investigate a general solution of equation (1) in a neighborhood of an ordinary point z = d, we initially make the following variable transformation:

(379) $w = z - d$, or

(380) $z = w + d$.

We will now substitute equation (380) into equation (1), apply Scheffé's criteria, and then investigate a solution of the resulting equation in a neighborhood of w = 0.

Substituting equation (380) into equation (1) gives us the following equation:

$$(381)\quad (w+d)(w+d-1)(w+d-a)(w+d-b)\frac{d^2u}{dw^2} + [A_1(w+d)^3 + A_2(w+d)^2 + A_3(w+d) + A_4]\frac{du}{dw} + [B_1(w+d)^2 + B_2(w+d) + B_3]u = 0,$$

where $A_i$, $(i = 1,2,3,4)$, and $B_j$, $(j = 1,2,3)$, are given by equations (2) - (8). Equation (381) may be written as follows:

$$(382)\quad p_2(w)\frac{d^2u}{dw^2} + p_1(w)\frac{du}{dw} + p_o(w)u = 0, \text{ where}$$

$$(383)\quad p_2(w) = (w+d)(w+d-1)(w+d-a)(w+d-b),$$

$$(384)\quad p_1(w) = A_1(w+d)^3 + A_2(w+d)^2 + A_3(w+d) + A_4, \text{ and}$$

$$(385)\quad p_o(w) = B_1(w+d)^2 + B_2(w+d) + B_3.$$

A general solution of non-confluent forms of equation (381) is contingent upon the assertion $d \neq 0, 1, a, b, \text{ or } \infty$.

We will now demonstrate that the restriction imposed by equation (14) is satisfied. Thus, upon applying Scheffé's criteria in the form given by equation (104) with $w = z - d$ and $j = 2$, gives us the following equation:

(386) $\quad p_2(w) = [S_2 - T_2 w^h] w^{2-m}.$

Equating the right member of this equation to the right member of equation (383) and assuming $T_2 = 0$, yields the following identity:

(387) $\quad S_2 w^{2-m} \equiv (w+d)(w+d-1)(w+d-a)(w+d-b).$

Evidently $2 - m = 4$, otherwise we are led to the obvious inconsistency $1 = 0$, when we equate like powers of w in the above identity. Hence $2 - m = 4$. Now equating the coefficients of like powers of w in both members of the above identity gives us the following set of equations:

(i) $\quad S_2 = 1,$

(ii) $\quad E_3 = 0,$

(iii) $\quad E_2 = 0,$

(iv) $\quad E_1 = 0,$ and

(v) $\quad d(d-1)(d-a)(d-b) = 0,$ where $E_i$ are the coefficients of $w^i$, $(i = 1,2,3)$, in equation (383). Now equation (v) requires that d be one of the finite regular singular points contrary to the assertion $d \neq 0, 1, a, b, \text{ or } \infty$. We therefore conclude that $T_2 \neq 0$.

Again in equation (386), we assume $S_2 = 0$, then equate the right member of the result to the right member of equation (383), thus obtaining the following identity:

(388) $-T_2 w^{h+2-m} \equiv (w+d)(w+d-1)(w+d-a)(w+d-b)$.

As before, we note that $h + 2 - m = 4$. Similarly, we obtain the following set of equations:

(i) $T_2 = -1$,

(ii) $E_3 = 0$,

(iii) $E_2 = 0$,

(iv) $E_1 = 0$, and

(v) $d(d-1)(d-a)(d-b) = 0$.

By previous arguments we see that equation (v) is not valid, therefore we conclude that $S_2 \neq 0$. Thus $S_2 T_2 \neq 0$, and the restriction given by equation (14) is satisfied.

We now set $j = 2$ in equation (104), then equate the result to the right member of equation (383) to obtain the following identity:

(389) $p_2(w) = S_2 w^{2-m} - T_2 w^{h+2-m} = (w+d)(w+d-1)(w+d-a)(w+d-b)$.

Now since h is a positive integer, then $h + 2 - m > 2 - m$, and $h + 2 - m = 4$. Otherwise $T_2 = 0$, which is contrary to previous observations. The equation $h + 2 - m = 4$ gives rise to the following cases which need investigation:

Case I: $h = 1$; $m = -1$,

Case II: $h = 2$; $m = 0$,

Case III: $h = 3$; $m = 1$, and

Case IV: $h = 4$; $m = 2$.

The above set of cases is exhaustive, and may be shown as follows: $h \geq 1$ is an integer, therefore $m \geq -1$ is an integer. Also $h > 4$ implies $m > 2$, hence $2 - m < 0$, which further implies from identity (389) that $S_2 = 0$. This is contrary to previous observations. We therefore conclude that the above set is exhaustive.

We observe in identity (389) that $2 - m = 0$, otherwise $d(d-1)(d-a)(d-b) = 0$, which violates our fundamental assertion $d \neq 0$, 1, a, b, or $\infty$. We then immediately conclude that Cases I, II, and III are beyond the scope of this study, and hence we investigate Case IV.

Case IV: $h = 4$, $m = 2$

Set $j = 2$, $h = 4$ and $m = 2$ in equation (104), then equate the result to the right member of equation (383) to get identity (389). From this identity we have the following equations upon equating like powers of w in both members:

(i) $T_2 = -1$,

(ii) $S_2 = d(d-1)(d-a)(d-b)$,

(iii) $E_3 = 0$,

(iv) $E_2 = 0$, and

(v) $E_1 = 0$.

Substituting equations (i) and (ii) into equation (386) gives us the following equation:

(390) $p_2(w) = w^4 + (d)(d-1)(d-a)(d-b)$.

Again in equation (104) we let $j = 1$, $h = 4$, and $m = 2$, then equate the result to the right member of equation (384) to obtain the following identity:

(391) $p_1(w) = S_1 w^{-1} - T_1 w^3 \equiv A_1(w+d)^3 + A_2(w+d)^2 + A_3(w+d) + A_4$.

Equating coefficients of like powers of w in the above identity gives us the following set of equations:

(i) $T_1 = -A_1$,

(ii) $S_1 = 0$,

(iii) $F_2 = 0$,

(iv) $F_1 = 0$, and

(v) $F_0 = 0$,

where $F_i$, $(i = 0,1,2)$, are the coefficients of $w^i$. Substituting equations (i) and (ii) into equation (391), and remembering from equation (2) that $A_1 = 1+\alpha+\beta$, gives us the following equation:

(392) $p_1(w) = (1+\alpha+\beta)w^3$.

Finally we compute $p_0(w)$ by letting $j = 0$, $h = 4$, and $m = 2$ in equation (104), then equating the result to the right member of equation (385), thus obtaining the following identity:

(393) $p_0(w) = S_0w^{-2} - T_0w^2 \equiv B_1(w+d)^2 + B_2(w+d) + B_3$.

Again we equate coefficients of like powers of $w$ in both members of the above identity to obtain the following set of equations:

(i) $T_0 = -B_1$,

(ii) $S_0 = 0$,

(iii) $2B_1d+B_2 = 0$, and

(iv) $B_1d^2 + B_2d+B_3 = 0$.

Again from equation (6), $B_1 = \alpha\beta$, hence equation (393) becomes:

(394) $p_0(w) = \alpha\beta w^2$.

Substituting equations (390), (392), and (394) into equation (382) yields the following differential equation:

$$(395)\quad [w^4 + d(d-1)(d-a)(d-b)]\frac{d^2u}{dw^2} + [(1+\alpha+\beta)w^3]\frac{du}{dw} + \alpha\beta w^2 u = 0.$$

Equation (395) is the equation we wish to solve.

Assume a solution of equation (395) in the form:

$$(396)\quad u(w) = c_0 w^{\lambda} + c_4 w^{\lambda+4} + \cdots + c_{4r} w^{\lambda+4r} + c_{4r+4} w^{\lambda+4r+4} + \cdots ;\ c_0 \neq 0.$$

Differentiating equation (396) with respect to w gives us:

$$(397)\quad u'(w) = c_0\lambda w^{\lambda-1} + c_4(\lambda+4)w^{\lambda+3} + \cdots + c_{4r}(\lambda+4r)w^{\lambda+4r-1} + c_{4r+4}(\lambda+4r+4)w^{\lambda+4r+3} + \cdots .$$

Differentiating equation (397) with respect to w yields:

$$(398)\quad u''(w) = c_0\lambda(\lambda-1)w^{\lambda-2} + c_4(\lambda+4)(\lambda+3)w^{\lambda+2} + \cdots + c_{4r}(\lambda+4r)(\lambda+4r-1)w^{\lambda+4r-2} + c_{4r+4}(\lambda+4r+4)(\lambda+4r+3)w^{\lambda+4r+2} + \cdots .$$

Substituting equations (396) - (398) into equation (395) gives us the following equation:

$$(399)\quad [w^4 + d(d-1)(d-a)(d-b)][c_0\lambda(\lambda-1)w^{\lambda-2} + c_4(\lambda+4)(\lambda+3)w^{\lambda+2} + \cdots + c_{4r}(\lambda+4r)(\lambda+4r-1)w^{\lambda+4r-2} + c_{4r+4}(\lambda+4r+4)(\lambda+4r+3)w^{\lambda+4r+2} + \cdots]$$
$$+ [(1+\alpha+\beta)w^3][c_0\lambda w^{\lambda-1} + c_4(\lambda+4)w^{\lambda+3} + \cdots + c_{4r}(\lambda+4r)w^{\lambda+4r-1} + c_{4r+4}(\lambda+4r+4)w^{\lambda+4r+3} + \cdots]$$
$$+ [\alpha\beta w^2][c_0 w^{\lambda} + c_4 w^{\lambda+4} + \cdots + c_{4r}w^{\lambda+4r} + c_{4r+4}w^{\lambda+4r+4} + \cdots] = 0.$$

We obtain the indicial equation by equating the coefficient of $w^{\lambda-2}$ to zero, thus:

$$(400)\quad c_0\lambda(\lambda-1)(d)(d-1)(d-a)(d-b) = 0,\ c_0 \neq 0.$$

Solving equation (400) yields:

$$(401)\quad \lambda_1 = 0, \text{ and } \lambda_2 = 1, \text{ since } d \neq 0, 1, a, b, \text{ or } \infty.$$

Similarly, to get the two term recurrence equation we equate the coefficient of $w^{\lambda+4r+2}$ to zero, thus:

$$(402)\quad c_{4r+4}(\lambda+4r+4)(\lambda+4r+3)(d)(d-1)(d-a)(d-b) + c_{4r}(\lambda+4r)(\lambda+4r-1) + c_{4r}(\lambda+4r)(1+\alpha+\beta) + c_{4r}(\alpha\beta) = 0.$$

Solving equation (402) for $c_{4r+4}$, yields:

$$c_{4r+4} = \frac{-[(\lambda+4r)(\lambda+4r-1) + (\lambda+4r)(1+\alpha+\beta) + \alpha\beta]}{(\lambda+4r+4)(\lambda+4r+3)(d)(d-1)(d-a)(d-b)} c_{4r}.$$

This equation may be simplified to the following equation:

$$(403) \quad c_{4r+4} = \frac{(\lambda+4r+\alpha)(\lambda+4r+\beta)\, c_{4r}}{(\lambda+4r+4)(\lambda+4r+3)(d)(1-d)(a-d)(b-d)}.$$

Let r = 0 in equation (403) to get:

$$(404) \quad c_4 = \frac{(\lambda+\alpha)(\lambda+\beta)\, c_0}{(\lambda+4)(\lambda+3)(d)(1-d)(a-d)(b-d)}.$$

Again, let r = 1 in equation (403) to get:

$$(405) \quad c_8 = \frac{(\lambda+\alpha+4)\,(\lambda+\beta+4)\, c_4}{(\lambda+4+4)(\lambda+4+3)(d)(1-d)(a-d)(b-d)}.$$

Substituting equation (404) into equation (405) gives us the following equation:

$$(406) \quad c_8 = \frac{(\lambda+\alpha)(\lambda+\alpha+4)(\lambda+\beta)(\lambda+\beta+4)\, c_0}{(\lambda+4)(\lambda+4+4)(\lambda+3)(\lambda+3+4)[d(1-d)(a-d)(b-d)]^2}.$$

To obtain a general term, we let r = n - 1 in equation (403) to get:

$$(407) \quad c_{4n} = \frac{(\lambda+\alpha)\cdots(\lambda+\alpha+4n-4)(\lambda+\beta)\cdots(\lambda+\beta+4n-4)[d(1-d)]^{-n} c_0}{(\lambda+4)\cdots(\lambda+4+4n-4)(\lambda+3)\cdots(\lambda+3+4n-4)[(a-d)(b-d)]^n}.$$

Dividing numerator and denominator of equation (407) by $4^{2n}$ gives us the desired expression for $c_{4n}$.

$$(408) \quad c_{4n} = \frac{(\frac{\lambda+\alpha}{4})\cdots(\frac{\lambda+\alpha}{4}+n-1)(\frac{\lambda+\beta}{4})\cdots(\frac{\lambda+\beta}{4}+n-1)[d(1-d)]^{-n} c_0}{(\frac{\lambda+4}{4})\cdots(\frac{\lambda+4}{4}+n-1)(\frac{\lambda+3}{4})\cdots(\frac{\lambda+3}{4}+n-1)[(a-d)(b-d)]^n}.$$

A general solution of equation (395) can now be written:

$$(409)\quad \bar{u}(w) = c_o w^{\lambda}\left[1 + \sum_{n=1}^{\infty} (c_{4n}/c_o) w^{4n}\right],$$

where $c_{4n}$ is given by equation (408) above. We now let $\lambda = 0$ from equation (401), then substitute equation (408) into equation (409) to get:

$$(410)\quad \bar{u}_1(w) = \bar{K}_1\left[1 + \sum_{n=1}^{\infty} \frac{(\frac{\alpha}{4})\cdots(\frac{\alpha}{4}+n-1)(\frac{\beta}{4})\cdots(\frac{\beta}{4}+n-1)w^{4n}}{n!(\frac{3}{4})\cdots(\frac{3}{4}+n-1)[d(1-d)(a-d)b-d)]^n}\right].$$

Comparing equation (410) with equation (84), we see that $\bar{u}_1(w)$ can be expressed in terms of the hypergeometric function if we first establish the following correspondences:

$a = \alpha/4$; $\quad c = 3/4$, and

$b = \beta/4$; $\quad z = w^4/[d(1-d)(a-d)(b-d)]$.

Equation (410) now becomes:

$$(411)\quad \bar{u}_1(w) = \bar{K}_1 F\left[\frac{\alpha}{4}, \frac{\beta}{4}; \frac{3}{4}; \frac{w^4}{d(1-d)(a-d)(b-d)}\right].$$

We now recall from equation (379) that $w = z - d$, hence if we let $\bar{u}_1(w) = \bar{u}_1(z-d) = u_1(z)$, then equation (411) becomes:

$$(412)\quad u_1(z) = \bar{K}_1 F\left[\frac{\alpha}{4}, \frac{\beta}{4}; \frac{3}{4}; \frac{(z-d)^4}{d(1-d)(a-d)(b-d)}\right].$$

We now let $\lambda = 1$ in equation (408), then substitute the result into equation (409) thus obtaining:

(413) $\bar{u}_2(w) = \bar{K}_2 w(1 + S)$, where

$$S = \sum_{n=1}^{\infty} \frac{(\frac{1+\alpha}{4})\cdots(\frac{1+\alpha}{4}+n-1)(\frac{1+\beta}{4})\cdots(\frac{1+\beta}{4}+n-1)w^{4n}}{n!(\frac{5}{4})\cdots(\frac{5}{4}+n-1)[d(1-d)(a-d)(b-d)]^n}.$$

We again compare equations (413) and (84) and note that equation (413) can be expressed in terms of the hypergeometric function, if we first establish the following correspondences:

$a = (1+\alpha)/4$; $\quad c = 5/4$, and

$b = (1+\beta)/4$; $\quad z = w^4/[d(1-d)(a-d)(b-d)]$.

Equation (413) then becomes:

(414) $$\bar{u}_2(w) = \bar{K}_2 wF\left[\frac{1+\alpha}{4}, \frac{1+\beta}{4}; \frac{5}{4}; \frac{w^4}{d(1-d)(a-d)(b-d)}\right].$$

Again we recall from equation (379) that $w = z - d$, hence $\bar{u}_2(w) = \bar{u}_2(z-d) = u_2(z)$. Equation (414) can finally be written:

(415) $$u_2(z) = \bar{K}_2(z-d)F\left[\frac{1+\alpha}{4}, \frac{1+\beta}{4}; \frac{5}{4}; \frac{(z-d)^4}{d(1-d)(a-d)(b-d)}\right].$$

A general solution of equation (395) is a linear combination of equations (412) and (415), thus:

$$(416)\quad u(z) = K_1 F\left[\frac{\alpha}{4}, \frac{\beta}{4}; \frac{3}{4}; \frac{(z-d)^4}{d(1-d)(a-d)(b-d)}\right]$$

$$+ K_2(z-d)F\left[\frac{1+\alpha}{4}, \frac{1+\beta}{4}; \frac{5}{4}; \frac{(z-d)^4}{d(1-d)(a-d)(b-d)}\right],$$

where $K_1$ and $K_2$ are arbitrary constants. Equation (416) is a general solution, in a neighborhood of an ordinary point $z = d$, of the equation which results from applying Scheffe's criteria to equation (1).

From the ratio test, we observe that equation (416) is absolutely convergent if

$$|z-d| < |d(1-d)(a-d)(b-d)|^{1/4}.$$

For the sake of convenience we will now rewrite equation (1).

$$(1)\quad z(z-1)(z-a)(z-b)\frac{d^2u}{dz^2} + [A_1z^3+A_2z^2+A_3z+A_4]\frac{du}{dz}$$

$$+ [B_1z^2+B_2z+B_3]u = 0.$$

This may be written in the form of equation (9), thus:

$$(9)\quad \sum_{j=0}^{2} p_j(z)\cdot\frac{d^ju}{dz^j} = 0, \text{ where}$$

$$(10)\quad p_2(z) = z(z-1)(z-a)(z-b),$$

$$(11)\quad p_1(z) = A_1z^3+A_2z^2+A_3z+A_4, \text{ and}$$

$$(12)\quad p_0(z) = B_1z^2+B_2z+B_3.$$

We now culminate this study by stating the following theorem which we have proved in this chapter:

Theorem: Let $p_j(z) = \sum_{i=0}^{j+2} C_{i,j} z^i$, where $C_{i,j}$, $(j = 0,1,2)$, are constants. A necessary condition for non-confluent forms of the differential equation $\sum_{j=0}^{2} p_j(z) \cdot \frac{d^j u}{dz^j} = 0$ to have solutions in a neighborhood of any point, which are expressible in terms of hypergeometric functions, is that Scheffé's criteria be applicable to $p_j(z)$.

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## BIOGRAPHICAL SKETCH

The writer of this dissertation was born in Okeechobee, Florida, April 16, 1927. He attended public schools in Polk and Osceola counties.

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This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of the committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

June 8, 1959

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